Subharmonic Orbits of a Strongly Nonlinear Oscillator Forced by Closely Spaced Harmonics

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We study asymptotically the family of subharmonic responses of an essentially nonlinear oscillator forced by two closely spaced harmonics. By expressing the original oscillator in action-angle form, we reduce it to a dynamical system with three frequencies (two fast and one slow), which is amenable to a singular perturbation analysis. We then restrict the dynamics in neighborhoods of resonance manifolds and perform local bifurcation analysis of the forced subharmonic orbits. We find increased complexity in the dynamics as the frequency detuning between the forcing harmonics decreases or as the order of a secondary resonance condition increases. Moreover, we validate our asymptotic results by comparing them to direct numerical simulations of the original dynamical system. The method developed in this work can be applied to study the dynamics of strongly nonlinear (nonlinearizable) oscillators forced by multiple closely spaced harmonics; in addition, the formulation can be extended to the case of transient excitations.
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1 Introduction

The majority of techniques for the asymptotic analysis of nonlinear oscillators relies on the use of harmonic generation [1–3]. In cases, however, where strong or even essential nonlinearities are encountered, that is, degenerate systems with nonlinearizble nonlinearities, new approaches should be adopted that take into account the nonlinearizable nature of the dynamics. Perturbation methods for strongly nonlinear systems have been developed [4–9] and were applied for studying the dynamics either in the absence of forcing (free responses) or when a single harmonic excitation is applied.

In this work, we develop an asymptotic methodology that is applicable to a general class of forced (or unforced) strongly nonlinear oscillators and apply it to the case of an essentially nonlinear (i.e., nonlinearizable) oscillator forced by two closely spaced harmonics. This later problem often arises in engineering practice, e.g., in problems of wave transmission in weakly coupled periodic structures or in problems of resonance capture in dynamical systems with multiharmonic excitations. The asymptotic analysis yields rich bifurcation structures in the forced dynamics, which become increasingly more complicated as the frequency detuning parameter denoting the difference between the frequencies of the forcing harmonics tends to zero or as the order of the resulting strongly nonlinear resonance interactions increases.

2 Nonlinear Boundary Value Problem Formulation

Consider the following strongly nonlinear oscillator forced by two harmonics with closely spaced frequencies:

$$w'' + \hat{C}w^3 = \hat{F}\sin t + \hat{F}\sin(1+\varepsilon^{1/4}\bar{B})t = \hat{F}[1+\cos(\varepsilon^{1/4}\bar{B}t)]\sin t + \hat{F}\sin(\varepsilon^{1/4}\bar{B}t)\cos t$$

$$w(0)=0, \quad w'(0)=W \tag{1}$$

where $\bar{B}$ represents the frequency detuning between the two harmonics of the excitation and the prime denotes differentiation with respect to the temporal variable t. As an initial step, we express the nonautonomous dynamical system (1) in the following autonomous form:

$$w' = z$$

$$z' = -\hat{C}w^3 + \hat{F}[1+\cos\theta_2]\sin\theta_1 + \hat{F}\sin\theta_2\cos\theta_1$$

$$\theta_1' = 1$$

$$\theta_2' = \varepsilon^{1/4}\bar{B}$$

$$w(0)=0, \quad z(0)=W \tag{2}$$

where $(w,z,\theta_1,\theta_2)\in R^2\times S^1\times S^1$.

To bring this system into a form better amenable to asymptotic analysis, we introduce the action-angle variables $(I,\phi)\in R^+\times S^1$ of the Hamiltonian oscillator obtained by setting $\hat{F}=0$ in Eq. (1) [10]. The explicit action-angle transformation applied to Eq. (1) is given by

$$w = \Lambda I^{1/3}\mathrm{cn}[2K(1/2)\phi/\pi, 1/2]$$

$$w' = -[\Lambda I^{1/3}\tilde{\omega}(I)2K(1/2)/\pi]\mathrm{sn}[2K(1/2)\phi/\pi,1/2]\,\mathrm{dn}[2K(1/2)\phi/\pi,1/2] \tag{3}$$

where w, w', I, and ϕ are functions of the independent temporal variable t, $\tilde{\omega}(I)=\Xi I^{1/3}$ is the instantaneous frequency of oscillation of the uncoupled nonlinear oscillator with $\Xi=[3\pi^4\hat{C}/8K^4(1/2)]^{1/3}$, $\Lambda=(4\hat{C})^{-1/6}[3\pi/K(1/2)]^{1/3}$, $K(1/2)$ the complete elliptic integral of the first kind, and sn and dn are Jacobi elliptic functions with modulus 1/2 [11]. Rescaling the forcing amplitude according to $\hat{F}=\varepsilon^{1/2}\hat{f}$, we then express the dynamical system (2) in the following *canonical form*:

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$$I' = -\varepsilon^{1/2}\hat{f}\frac{3I^{1/3}\pi([1+\cos\,\theta_2]\sin\,\theta_1 + \sin\,\theta_2\,\cos\,\theta_1)}{2K(1/2)\Lambda\Xi}\frac{\text{sndn}}{\text{cn}^4 + 2\text{sn}^2\text{dn}^2}$$

$$\equiv \varepsilon^{1/2}\tilde{f}_1(I,\phi,\theta_1,\theta_2)$$

$$\phi' = \tilde{\omega}(I) - \varepsilon^{1/2}\hat{f}\frac{\pi^2 I^{-2/3}([1+\cos\,\theta_2]\sin\,\theta_1 + \sin\,\theta_2\,\cos\,\theta_1)}{4K^2(1/2)\Lambda\Xi}$$

$$\times\frac{\text{cn}}{\text{cn}^4 + 2\text{sn}^2\text{dn}^2} \equiv \tilde{\omega}(I) + \varepsilon^{1/2}\tilde{f}_2(I,\phi,\theta_1,\theta_2)$$

$$\theta_1' = 1$$

$$\theta_2' = \varepsilon^{1/4}\bar{B} \qquad (4a)$$

with initial conditions

$$I^{2/3}(0) = \frac{-W\pi}{2^{1/2}\Lambda\Xi K(1/2)} > 0, \quad \phi(0) = 0 \qquad (4b)$$

In the relations above, sn, cn, and dn are Jacobi elliptic functions with argument $[2K(1/2)(\phi+\pi/2)/\pi, 1/2]$. For more details of this computation, we refer to the work of Vakakis and Gendelman [12].

The dynamics of system (4*a*) takes place on the three-torus $(I,\phi,\theta_1,\theta_2) \in R^+ \times S^1 \times S^1 \times S^1$. Moreover, this dynamical system possesses two *fast frequencies* equal to $\tilde{\omega}(I)+O(\varepsilon^{1/2})$ and 1, respectively, and a *slow frequency* equal to $\varepsilon^{1/4}\bar{B}$. It follows that we may eliminate the independent slow frequency and reduce the dynamics to the following dynamical system on a two-torus:

$$I' = \varepsilon^{1/2}\tilde{f}_1(I,\phi,\theta,\varepsilon^{1/4}\bar{B}\theta)$$

$$\phi' = \tilde{\omega}(I) - \varepsilon^{1/2}\tilde{f}_2(I,\phi,\theta,\varepsilon^{1/4}\bar{B}\theta)$$

$$\theta' = 1 \qquad (5)$$

where we introduced the renaming $\theta_1 \to \theta$. The dynamical system (5) represents a *global model* of the dynamics since it is valid for arbitrary values of the action-angle variables.

We wish to study the structure of the subharmonic orbits of the dynamical system (5) in the limit of small ε. To this end, we need to construct a countable infinity of *local models* by considering the unperturbed system with $\varepsilon=0$ and imposing *internal resonance conditions* to the two fast frequencies. For example, considering the subharmonic orbits of system (5) (or, equivalently, of system (4*a*)) satisfying an $(m:n)$ ratio between the two fast frequencies, we impose the following condition, which computed the corresponding value of the action variable:

$$m\tilde{\omega}(I) - n = 0 \Rightarrow I = \left(\frac{n}{m\Xi}\right)^3 \equiv I^{(m/n)}, \quad n,m \in N^+ \qquad (6)$$

This relation couples the two fast frequencies of the reduced problem and defines the $(m:n)$ *resonance manifold* for the dynamics. We then study the dynamics in the neighborhood of this resonant manifold by defining the new dependent variable $\gamma=m\phi-n\theta$, which denotes deviations of the two fast angles from values satisfying the condition of internal resonance when the $O(\varepsilon^{1/2})$ perturbation terms are taken into account. Introducing the change in variables $(\phi,\theta)\to(\psi,\theta)$ and changing the independent variable from t to θ (this is permissible since we can solve the last of Eq. (5) as, $\theta=t+\theta_0$, so that θ is a timelike fast angle—for simplicity, we take $\theta_0=0$), we may further reduce the dynamical system (5) to the following form:

$$I'(\theta) = \varepsilon^{1/2}f_1(I,\gamma,\theta,\varepsilon^{1/4}\bar{B}\theta)$$

$$\gamma'(\theta) = [m\tilde{\omega}(I) - n] + \varepsilon^{1/2}f_2(I,\gamma,\theta,\varepsilon^{1/4}\bar{B}\theta) \qquad (7)$$

where differentiations are carried out with respect to θ, and the right-hand-side terms are defined according to $f_i(I,\gamma,\theta,\varepsilon^{1/4}\bar{B}\theta) = \hat{f}_i(I,\phi=(\gamma+n\theta)/m,\theta,\varepsilon^{1/4}\bar{B}\theta)$, $i=1,2$; note that in these definitions, the integers n and m (used in the internal resonance condition (6)) enter implicitly. We now introduce the following action coordinate change $I(\theta)=I^{(m/n)}+\varepsilon^{1/4}\xi(\theta)$ in order to study the dynamics of the reduced system close to the $(m:n)$ resonance manifold. Then, the following *local reduced order* model is obtained, which in contrast to the global model (5) is valid only in the neighborhood of the $(m:n)$ resonance manifold,

$$\xi'(\theta) = \varepsilon^{1/4}f_1(I^{(m/n)},\gamma,\theta,\varepsilon^{1/4}\bar{B}\theta) + \varepsilon^{1/2}\frac{\partial f_1(I^{(m/n)},\gamma,\theta,\varepsilon^{1/4}\bar{B}\theta)}{\partial I}$$

$$+ O(\varepsilon^{3/4})$$

$$\gamma'(\theta) = \varepsilon^{1/4}m\xi(\theta)\tilde{\omega}'(I^{(m/n)}) + \varepsilon^{1/2}\{(m/2)\tilde{\omega}''(I^{(m/n)})\xi^2(\theta)$$

$$+ mf_2(I^{(m/n)},\gamma,\theta,\varepsilon^{1/4}\bar{B}\theta)\} + O(\varepsilon^{3/4}) \qquad (8)$$

where the prime denotes differentiation of a function with respect to its argument.

The local dynamical system (8) possesses a resonance capture topology [12–15] in the $O(\varepsilon^{1/4})$ neighborhood of the $(m:n)$ resonance manifold, with the main difference from previous studies being the periodic dependence of the terms f_1 and f_2 on the slow angle $\varepsilon^{1/4}\bar{B}\theta$. Moreover, system (8) is in standard form for applying averaging [3] or for applying a singular perturbation analysis such as the method of multiple scales [1]. In what follows, we will adopt the later approach by introducing the new timelike scales $\delta=\theta$ and $\eta=\varepsilon^{1/4}\theta$ and express the solution of the local model in the series form:

$$\xi(\theta) = \xi(\delta,\eta) = \xi_0(\delta,\eta) + \varepsilon^{1/4}\xi_1(\delta,\eta) + \cdots$$

$$\gamma(\theta) = \gamma(\delta,\eta) = \gamma_0(\delta,\eta) + \varepsilon^{1/4}\gamma_1(\delta,\eta) + \cdots \qquad (9)$$

Substituting into the local model (8), we derive a hierarchy of subproblems at different orders of approximation, which we analyze separately.

The $O(1)$ subproblem is solved as follows:

$$\frac{\partial\xi_0}{\partial\delta} = 0 \Rightarrow \xi_0(\delta,\eta) = C_1(\eta)$$

$$\frac{\partial\gamma_0}{\partial\delta} = 0 \Rightarrow \gamma_0(\delta,\eta) = C_2(\eta) \qquad (10a)$$

where $C_1(\eta)$ and $C_2(\eta)$ are functions of the slow scale, which are determined by considering terms in the next order of approximation.

Proceeding to the $O(\varepsilon^{1/4})$ subproblem, this is given by

$$\frac{\partial\xi_1}{\partial\delta} = f_1(I^{(m/n)},C_2(\eta),\delta,\bar{B}\eta) - C_1'(\eta)$$

$$\frac{\partial\gamma_1}{\partial\delta} = m\tilde{\omega}'(I^{(m/n)})C_1(\eta) - C_2'(\eta) \qquad (10b)$$

Terms that are constant in terms of the fast scale δ represents secular terms in the above system and need to be eliminated in order to obtain uniformly valid solutions as $\delta\to\infty$. The conditions for eliminating these secular terms lead to the following equations, determining the functions $C_1(\eta)$ and $C_2(\eta)$ of the $O(1)$ approximation:

$$-C_1'(\eta) + \frac{1}{T}\int_0^T f_1(I^{(m/n)},C_2(\eta),\delta,\bar{B}\eta)d\delta = 0$$

$$- C_2'(\eta) + m\tilde{\omega}'(I^{(m/n)})C_1(\eta) = 0 \quad (11a)$$

where T is the minimal period of the function f_1 with respect to the fast scale δ and the prime denotes differentiation with respect to the slow angle η. These equations can be combined to a single second order equation in terms of $C_2(\eta)$ as follows:

$$C_2''(\eta) - \frac{m\tilde{\omega}'(I^{(m/n)})}{T}\int_0^T f_1(I^{(m/n)}, C_2(\eta), \delta, \bar{B}\eta)d\delta = 0 \quad (11b)$$

This equation provides the leading order approximation to the resonance capture dynamics in the $O(\varepsilon^{1/4})$ neighborhood of the $(m:n)$ resonance manifold (formulated in full generality in the work of Verhulst [16]; Eq. 11.4, Sec. 11.6). We note that in the limit $\bar{B}\to 0$, i.e., of no slow frequency dependence in the nonhomogeneous part of the oscillator (1), relation (11b) provides the well-known pendulum equation that governs resonance capture dynamics close to an invariant manifold for a system possessing only two "fast" frequencies; this topic has been studied in previous works [13,12,15]. We wish to study how the resonance capture topology is perturbed when a third slow frequency is added to the dynamics.

To perform a more detailed analysis of the resonance capture dynamics in the problem under consideration, we set $m=n=1$ (for simplicity) and consider the dynamics in the neighborhood of the 1:1 resonance manifold. For the oscillator (1), the stable invariant submanifolds of this specific manifold are expected to possess the broadest domains of attraction and hence to most strongly influence the forced dynamics compared with other members of the countable infinity of $(m:n)$ resonance manifolds [15]. For $m=n=1$, Eq. (11b) is expressed as

$$C_2''(\eta) + G\sin\bar{B}\eta\cos C_2(\eta) - G[1+\cos\bar{B}\eta]\sin C_2(\eta) = 0 \quad (12)$$

where

$$G = -\hat{f}\tilde{\omega}'(I^{(1/1)})\frac{3\pi 0.4045 I^{(1/1)1/3}}{2K(1/2)\Lambda\Xi}$$

The periodic orbits of system (12) in terms of the slow time scale η will be numerically computed utilizing the method of nonsmooth coordinate transformations first developed by Pilipchuk [17,18] and then applied to strongly nonlinear oscillators by Pilipchuk et al. [19]. To apply the method, we express the sought periodic solutions in terms of two nonsmooth variables $\tau(\hat{\eta})$ and $e(\hat{\eta})$ defined as

$$\tau(\hat{\eta}) = \frac{2}{\pi}\sin^{-1}\left(\sin\frac{\pi\hat{\eta}}{2}\right), \quad e(\hat{\eta}) = \tau'(u)$$

where $\hat{\eta}=\eta/a$ and a is the quarter-period of the solution (confer Fig. 1). In addition, we impose the secondary resonance condition $\bar{B}=k\pi/2a,\ k=1,2,\ldots$, which is necessary for the realization of periodic orbits in Eq. (12). Then, we express the solution as

$$C_2(\hat{\eta}) = C_2[\tau(\hat{\eta}), e(\hat{\eta})] = X(\tau) + e\Psi(\tau) \quad (13)$$

substitute this expression into Eq. (12) and set separately equal to zero the components that depend or not on the nonsmooth variable e. Then, we obtain the following nonlinear boundary value problems (NLBVPs) over the finite interval $-1\le\tau\le 1$: $k=2p+1,\ p\in N^+$,

$$X'' + Ga^2\sin[(2p+1)\pi\tau/2]\cos X\cos Y - Ga^2\sin X\cos Y - Ga^2\cos[(2p+1)\pi\tau/2]\cos X\sin Y = 0$$

$$Y'' - Ga^2\sin[(2p+1)\pi\tau/2]\sin X\sin Y - Ga^2\cos X\sin Y - Ga^2\cos[(2p+1)\pi\tau/2]\sin X\cos Y = 0$$

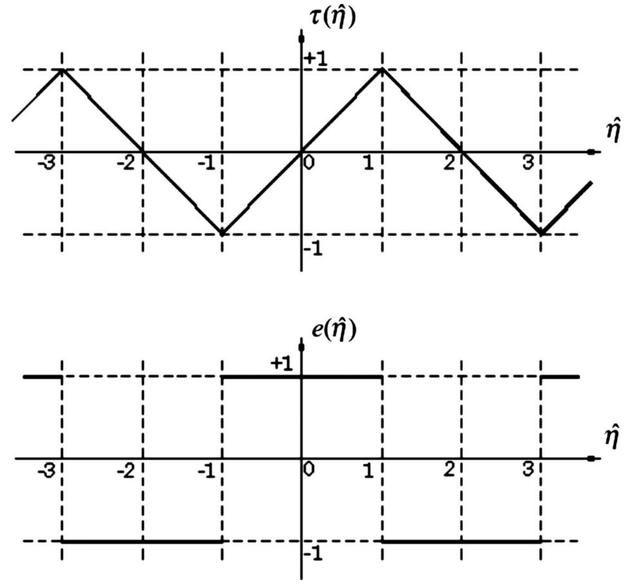

Fig. 1 Nonsmooth variables $\tau(\hat{\eta})$ and $e(\hat{\eta})$

$$X'(\pm 1) = 0, \quad Y(\pm 1) = 0 \quad (14a)$$

$k=2p,\ p\in N^+$,

$$X'' - Ga^2\sin p\pi\tau\sin X\sin Y - Ga^2\sin X\cos Y - Ga^2\cos p\pi\tau\sin X\cos Y = 0$$

$$Y'' + Ga^2\sin p\pi\tau\cos X\cos Y - Ga^2(1+\cos p\pi\tau)\cos X\sin Y = 0$$

$$X'(\pm 1) = 0, \quad Y(\pm 1) = 0 \quad (14b)$$

where differentiations are carried out with respect to the nonsmooth variable τ. The boundary conditions in Eqs. (14a) and (14b) are derived by imposing "smoothening conditions" [19] in order to eliminate singular terms from the transformed equations such as terms proportional to $e'(\hat{\eta})=\tau''(\hat{\eta})=2\sum_{k=-\infty}^{\infty}[\delta(\hat{\eta}+1-4k) - \delta(\hat{\eta}-1-4k)]$.

We note that the solutions of the NBVPs (14a) and (14b) correspond to first-order approximations to the slow-varying solutions of the reduced local model (7) under the condition of internal resonance (6) with $m=n=1$. Considering the action-angle transformation (3), the first-order approximation of the subharmonic response of the strongly nonlinear forced oscillator (1) is then expressed as

$$w(t) = \Lambda[I^{(m/n)} + \varepsilon^{1/4}C_1(\varepsilon^{1/4}t) + O(\varepsilon^{1/2})]^{1/3}\mathrm{cn}\left\{\frac{2K(1/2)}{\pi}\left[\frac{nt + C_2(\varepsilon^{1/4}t) + O(\varepsilon^{1/4})}{m} + \frac{\pi}{2}\right], 1/2\right\} \quad (15a)$$

with $m=n=1$ and satisfying the initial conditions

$$w(0) = 0 \ \text{ and } \ w'(0) = \Lambda[I^{(m/n)} + \varepsilon^{1/4}C_1(0) + O(\varepsilon^{1/2})]^{1/3} \equiv W \quad (15b)$$

It follows that for a general case of $(m:n)$ internal resonance the forced subharmonic response of Eq. (1) is in the form of a slowly modulated signal possessing two fast frequencies equal to 1 and n/m, respectively, and a slow frequency equal to $\varepsilon^{1/4}\bar{B}$. Typically, this is a *quasiperiodic* response, unless the slow frequency is in rational relation with respect to either one of the fast frequencies, in which case the response is *periodic*.

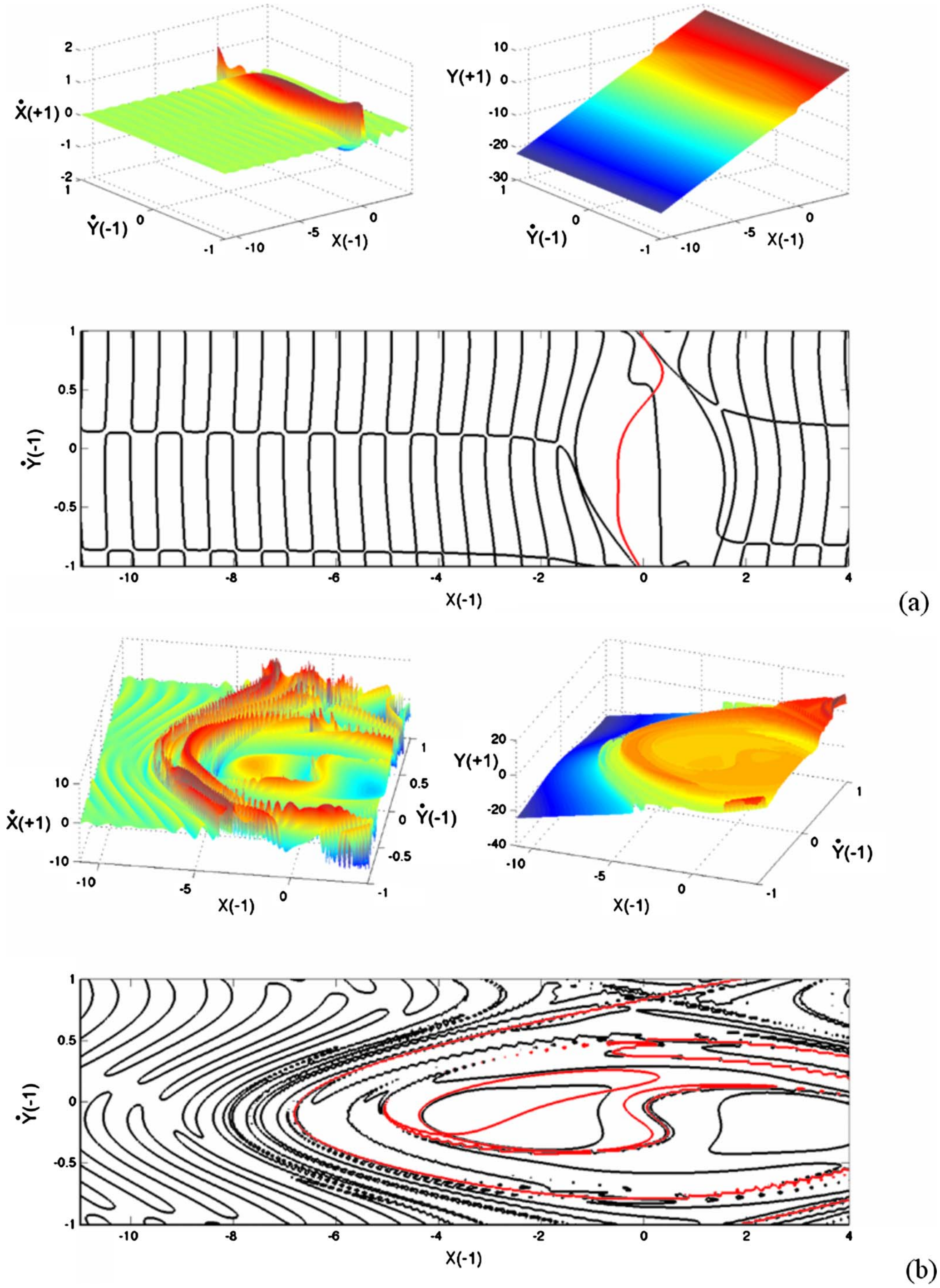

Fig. 2 Surfaces $S_{X'}(p_{Y'},p_X)$ and $S_Y(p_{Y'},p_X)$ of the NLBVP (14*a*) for $k=1$ and (*a*) $\bar{B}=0.671$ and (*b*) $\bar{B}=0.101$; the lower plots show the zero contours of $S_{X'}(p_{Y'},p_X)$ (black lines) and $S_Y(p_{Y'},p_X)$ (red lines for the online version and grey for the printed version)

In the next section, we study in detail the solutions (and their bifurcations) of the NLBVPs (14*a*) and (14*b*) (corresponding to 1:1 internal resonance) and relate them to the subharmonic forced responses of the original problem (1).

3 Forced Subharmonic Responses

In this section, we study the topology of the subharmonic orbits of Eq. (1) under condition of 1:1 internal resonance (i.e., $m=n=1$) by numerically solving the NLBVPs (14*a*) and (14*b*). As mentioned in the last section, in the limit $\bar{B}\to 0$ (i.e., when the frequency detuning between the two forcing harmonics tends to zero), the dynamical system (12) takes the form of an unforced pendulum equation; it is well known that this system possesses a homoclinic loop that surrounds an infinite family of periodic orbits, which represents a classical resonance capture topology in the dynamics of a single-degree-of-freedom nonlinear oscillator forced by a single harmonic excitation [13]. We wish to study how this resonance capture topology is perturbed as the frequency detuning parameter $\bar{B}$ increases from zero.

It turns out that solving the NLBVPs (14*a*) and (14*b*) is not a

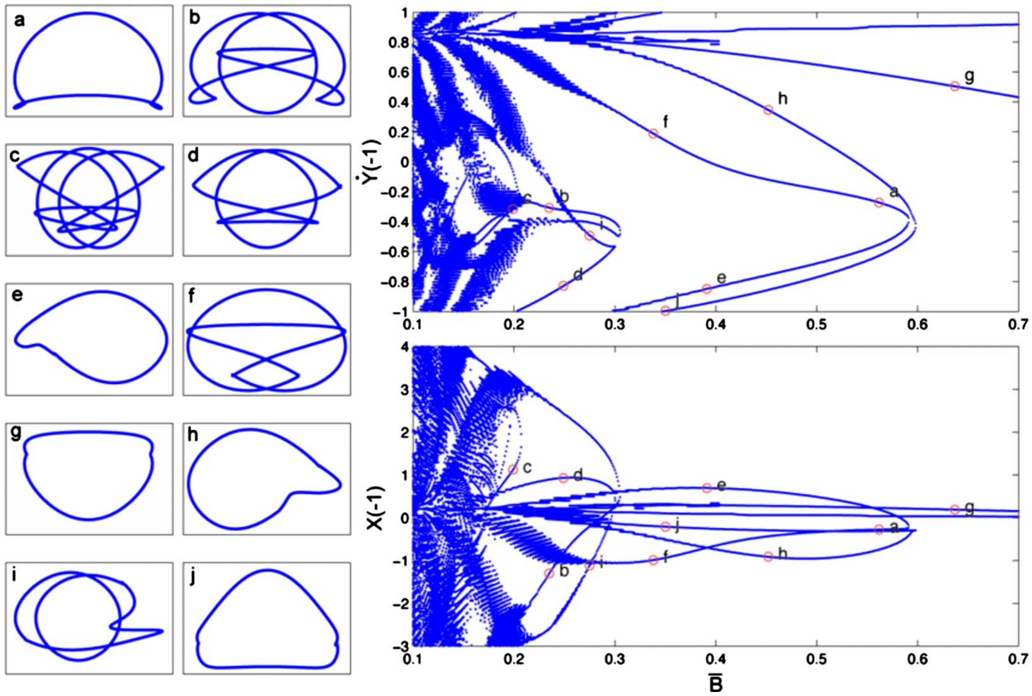

Fig. 3 Bifurcation diagrams of the solutions of the NLBVP (14*a*) for 1:1 resonance and *k*=1; on the left, we depict some characteristic periodic orbits of (12) corresponding to points marked in the bifurcation diagram

simple task due to the occurrence of very complex bifurcations in the dynamics of Eq. (12) for varying parameter $\bar{B}$. To address this problem, we employed a special numerical technique by treating the coupled equations in Eq. (14*a*) or Eq. (14*b*) as *initial value problems (IVPs)* with initial values prescribed at $\tau=-1$ given by

$$X(-1) \equiv p_X, \quad Y'(-1) \equiv p_{Y'}, \quad X'(-1)=0, \quad Y(-1)=0 \tag{16}$$

By numerically solving the IVPs, we obtain the values $X'(+1)$ and $Y(+1)$ as functions of the initial conditions p_X and $p_{Y'}$. Therefore, we can numerically generate the following two two-dimensional surfaces:

$$X'(+1) \equiv S_{X'}(p_{Y'},p_X)$$

$$Y(+1) \equiv S_Y(p_{Y'},p_X)$$

Clearly, the solutions of NLBVP (14*a*) or NLBVP (14*b*) are the points $(p_{Y'},p_X)$ satisfying the set of nonlinear relations

$$S_{X'}(p_{Y'},p_X)=0$$

$$S_Y(p_{Y'},p_X)=0$$

From a geometrical point of view, each of the relations above defines a one-dimensional line, so the intersections of these lines define the sought solutions. From a computational point of view, in order to compute these points, first, we compute the zero contours of each of the two-dimensional surfaces, i.e., the sets:

$$C_{X'}=\{(p_{Y'},p_X) \in \mathbb{R}\times S^1 | S_{X'}(p_{Y'},p_X)=0\}$$

$$C_Y=\{(p_{Y'},p_X) \in \mathbb{R}\times S^1 | S_Y(p_{Y'},p_X)=0\}$$

Then, we compute the solutions of the NLBVP (14*a*) or NLBVP (14*b*) by calculating the intersections of the above curves

$$\sigma=\{(p_{Y'},p_X) \in \mathbb{R}\times S^1 | S_Y(p_{Y'},p_X)=S_{X'}(p_{Y'},p_X)=0\}.$$

In this way, we obtain the solutions (and their bifurcations) of the NLBVP as solutions of the sets of Eq. (14*a*) or Eq. (14*b*), subject to the initial conditions (16) with $(p_{Y'},p_X)\in\sigma$.

Note that the above method has the added advantage to predict the existence of periodic orbits without having to determine the corresponding pair $(p_{Y'},p_X)$ with high accuracy. This is because the existence of a solution of the NLBVP is guaranteed by the existence of an intersection of the sets $C_{X'}$ and C_Y, even though the intersection point does not need to be computed with high accuracy. This is particularly important for determining unstable solutions, whose computation is very sensitive to even small errors in the estimates of $(p_{Y'},p_X)$.

This last feature of the outlined methodology plays a critical role in the following computations since it will be shown that even though for relatively large values of the detuning frequency parameter $\bar{B}$, the surfaces $S_{X'}(p_{Y'},p_X),S_Y(p_{Y'},p_X)$ are sufficiently smooth; this behavior does not persist when $\bar{B}$ tends toward zero. In this case, it will be shown that these surfaces have highly oscillating regions and the curves $C_{X'}$ and C_Y have very complex topologies, reflecting the complex bifurcation structure leading to an infinite family of solutions (or equivalently periodic orbits) as $\bar{B}\to 0$; i.e., as the classical topology of resonance capture is reached.

In Figs. 2(*a*) and 2(*b*), we present the surfaces $S_{X'}(p_{Y'},p_X),S_Y(p_{Y'},p_X)$ (upper plots) along with their zero contours (lower plots) for the NLBVP (14*a*) with $k=1$, parameters $\hat{C}=1.0$, $\varepsilon=0.001$, $\omega_1=1.0$, and $\hat{f}=1.0$, and two values of the frequency detuning parameter $\bar{B}$. In the lower plots, the black lines represent the zero contours of the surface $S_{X'}(p_{Y'},p_X)$, whereas the red curves correspond to the zero contours of the surface $S_Y(p_{Y'},p_X)$. The intersections of the black and red lines (grey lines for the printed version) provide the special set of points σ that are

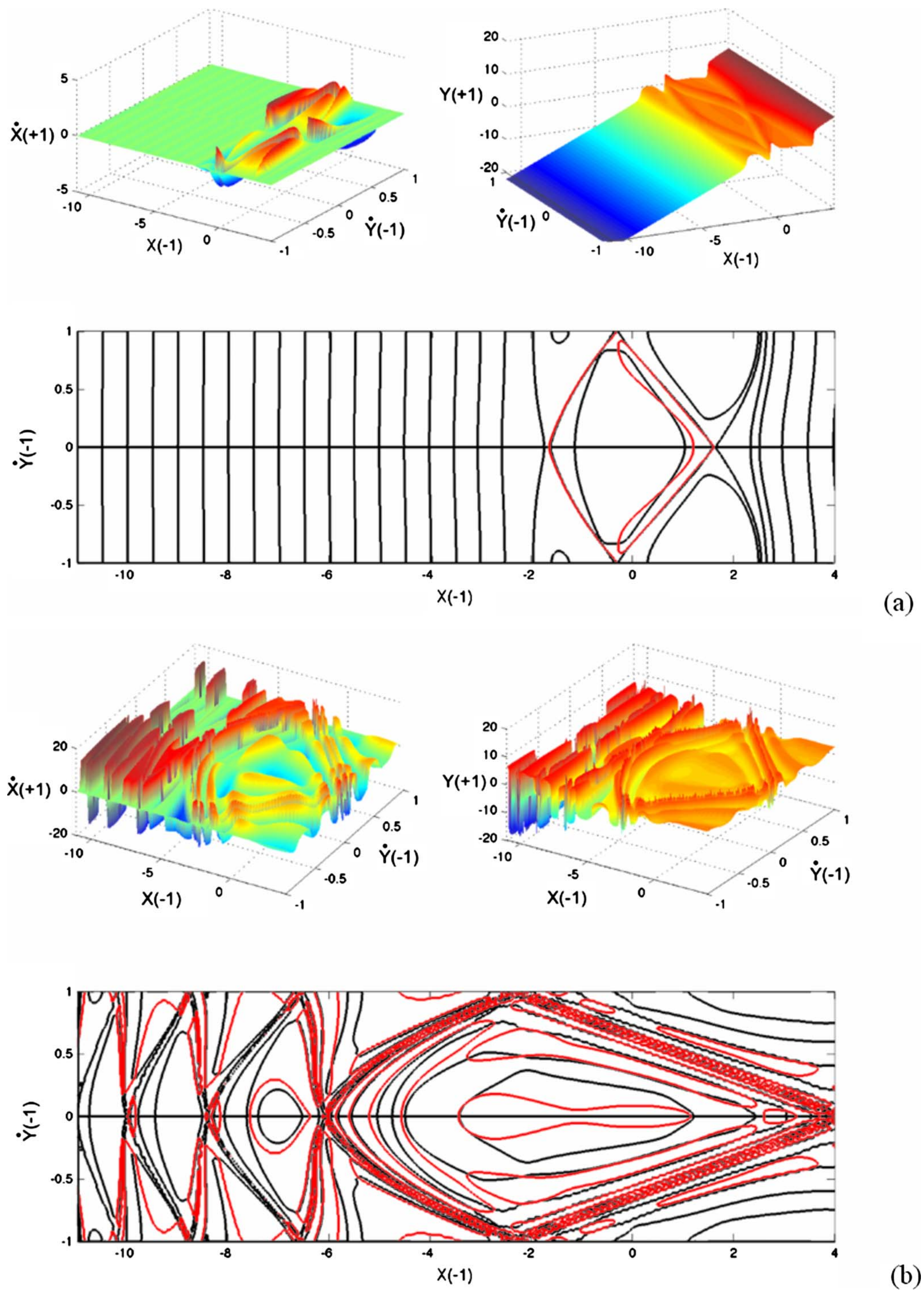

Fig. 4 Surfaces $S_{X'}(p_{Y'},p_X)$ and $S_Y(p_{Y'},p_X)$ of the NLBVP (14*b*) for *k*=2 and (*a*) $\bar{B}$=0.671 and (*b*) $\bar{B}$=0.101; the lower plots show the zero contours of $S_{X'}(p_{Y'},p_X)$ (black lines) and $S_Y(p_{Y'},p_X)$ (red lines for the online version and grey lines for the printed version)

solutions of the NLBVP (14*a*) (or, equivalently, periodic solutions with respect to the slow angle of the problem (12)).

For the larger value of $\bar{B}$ (Fig. 2(*a*)), we observe that σ consists of two intersection points corresponding to two periodic orbits of Eq. (12). Hence, we find that for relatively large values of $\bar{B}$ the infinity of periodic orbits encountered in the classical resonance capture topology is restricted to only 2. Clearly, as $\bar{B}$ decreases, we anticipate that the topology of the two surfaces will become more complex in order to yield through their intersections a continuously increasing number of bifurcating periodic orbits and, ultimately, an infinity of periodic orbits as the limit $\bar{B}\rightarrow 0$ is reached. This is confirmed in the lower plot of Fig. 2(*b*), where we note that the zero contours tend to align with each other, creating a large number of intersection points (solutions of the NLBVP). It follows that with the above representation of solutions, we are able to describe the transition from the finite number of periodic orbits of Eq. (12) to the infinite family of periodic solutions as

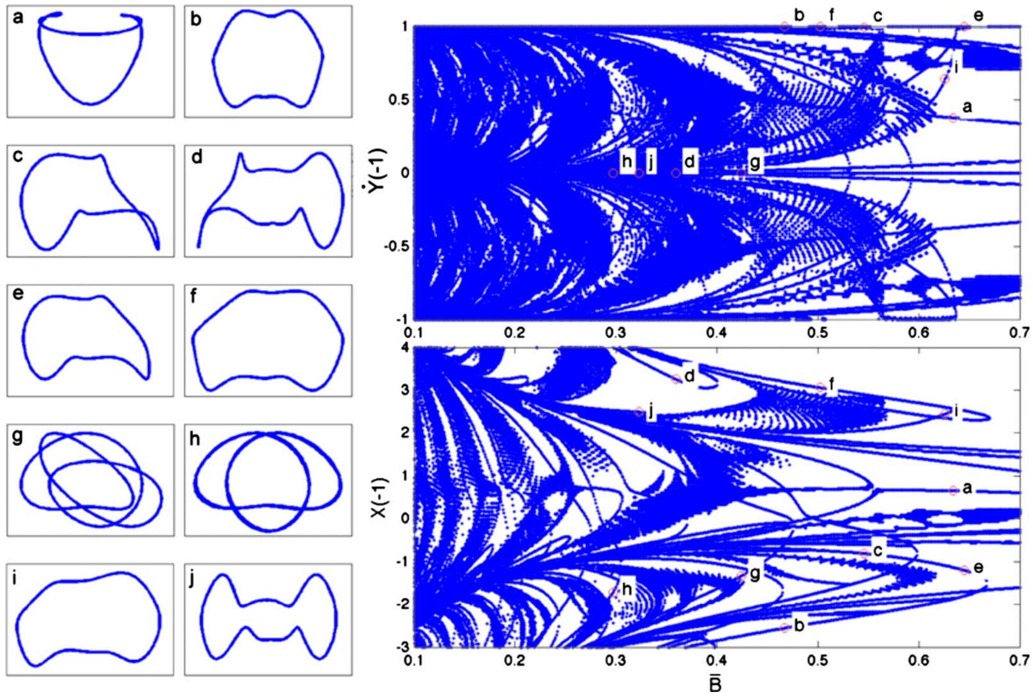

Fig. 5 Bifurcation diagrams of the solutions of the NLBVP (14*b*) for 1:1 resonance and $k=2$; on the left, we depict some characteristic periodic orbits of Eq. (12) corresponding to points marked in the bifurcation diagram

$\bar{B}\to 0$. This transition is better illustrated in the bifurcation diagram depicted in Fig. 3, where the values of $X(-1)$ and $Y'(-1)$ corresponding to solutions of the NLBVP are presented in a bifurcation diagram with respect to $\bar{B}$. In the same figure, we depict some characteristic periodic orbits of system (12) in the $(C_2(\eta), C_2'(\eta))$ plane, each one corresponding to a pair of $(X(-1), Y'(-1))$ marked in the bifurcation diagram.

In Fig. 4, we depict the surfaces $S_{X'}(p_{Y'}, p_X), S_Y(p_{Y'}, p_X)$ and their zero contours for $k=2$ and two values of the frequency detuning parameter. Moreover, in Fig. 5, the corresponding bifurcation diagrams are shown, where we observe that the complexity of the topology of the subharmonic orbits is more complex compare with the case $k=1$. This enhancement of complexity is also manifested in Fig. 6 where the zero contour curves for the cases $k=3$, $k=4$, and frequency detuning parameter $\bar{B}=0.671$ are presented.

Finally, in order to test the validity of the derived asymptotic approximations, we performed an additional numerical study where the subharmonic solutions obtained by numerically solving the NLBVPs (14*a*) and (14*b*) were compared with direct numerical simulations of the original strongly nonlinear oscillator (Eq. (1)) (or, equivalently, Eq. (3)) subject to the theoretically predicted initial conditions. In Fig. 7, we present two different periodic responses of the system for resonance parameter $k=1$, corresponding to periodic orbits (a) and (c) in the bifurcation diagram of Fig. 3. Agreement between the results of the asymptotic analysis and the direct numerical simulations is noted, validating our asymptotic approach. In Fig. 8, we compare two additional subharmonic responses of the oscillator for $k=2$, corresponding to periodic orbits (b) and (h) in the bifurcation diagram of Fig. 5. Again, the comparisons validate the asymptotic results.

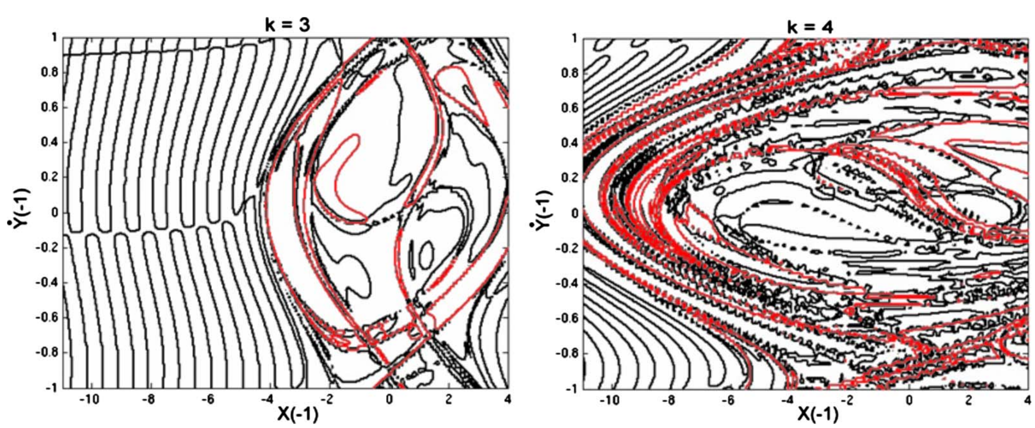

Fig. 6 Surfaces $S_{X'}(p_{Y'}, p_X)$ and $S_Y(p_{Y'}, p_X)$ of the NLBVPs (14*a*), $k=3$, and (14*b*), $k=4$, for $\bar{B}=0.671$; the zero contours of $S_{X'}(p_{Y'}, p_X)$ are denoted by black lines and of $S_Y(p_{Y'}, p_X)$ by red lines for the online version and grey lines for the printed version

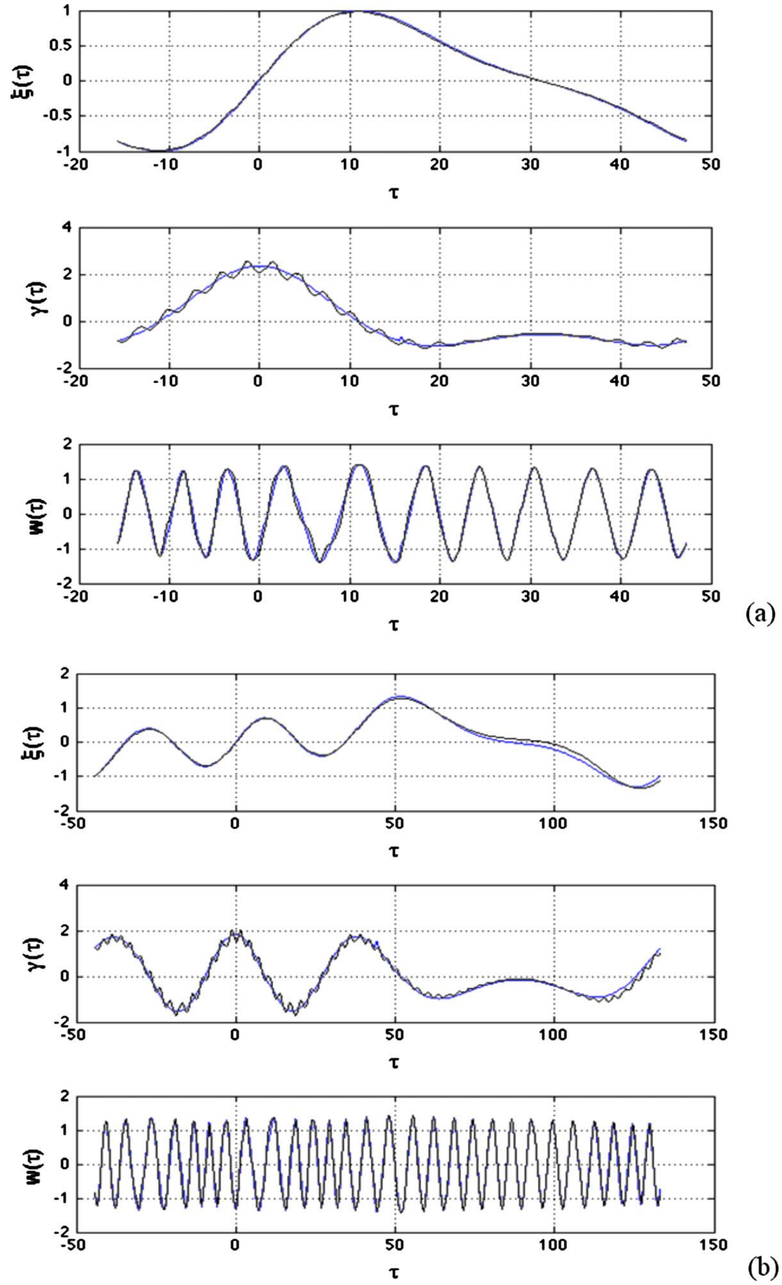

Fig. 7 Subharmonic responses computed by solving the NLBVP (14*a*) for *k*=1 (blue curves) and the original forced strongly nonlinear oscillator (1) (black curves) for $\varepsilon=0.001$; the subharmonic responses depicted in (*a*) and (*b*) correspond to the periodic orbits (*a*) and (*c*) in the bifurcation diagram of Fig. 3, respectively

4 Concluding Remarks

We developed an asymptotic method for computing the subharmonic responses of a strongly nonlinear oscillator forced by two closely spaced harmonics. The method is based on action-angle transformation of the original differential equation of motion, which brings the system into canonical form, which can be asymptotically analyzed by methods from singular perturbation methods. We emphasize that our analysis is not based on the use of harmonic (i.e., linearized) generating functions as most current techniques do; in fact, an approach based on linearization would not be applicable in the system under consideration as it possesses an essentially nonlinear (i.e., nonlinearizable) stiffness element. Comparisons of the asymptotic results to direct numerical simulations proved the validity of our approach.

A bifurcation study was carried out for the case of 1:1 resonance between the frequency of oscillation of the oscillator and one of the forcing harmonics. Our analysis indicates that the forced strongly nonlinear oscillator possesses very complex dynamics. Indeed, the bifurcation diagrams (which were constructed using the frequency detuning parameter between the forcing harmonics as bifurcation parameter) indicated that the forced dynamics becomes increasingly more complex as the frequency detuning

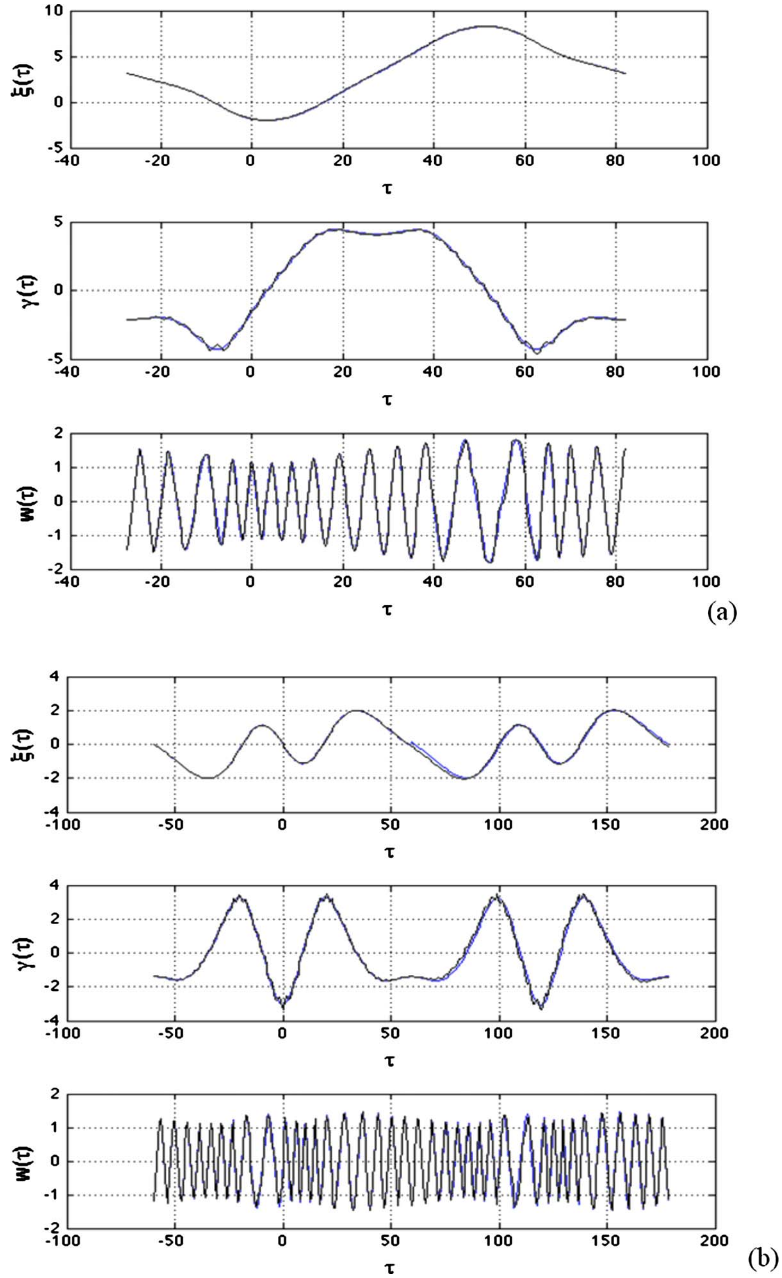

Fig. 8 Subharmonic responses computed by solving the NLBVP (14*b*) for $k=2$ (blue curves) and the original forced strongly nonlinear oscillator (1) (black curves) for $\varepsilon=0.001$; the subharmonic responses depicted in (*a*) and (*b*) correspond to the periodic orbits (*b*) and (*h*) in the bifurcation diagram of Fig. 5, respectively

parameter tends to zero or as the order of the secondary resonance k increases. This is a manifestation of the fact that in the limit of zero detuning the strongly nonlinear oscillator is forced by a single harmonic excitation and exhibits chaotic responses. Depending on the frequency detuning $\varepsilon^{1/4}\bar{B}$, the subharmonic response of the oscillator is periodic or quasiperiodic.

The developed action-angle based methodology can be extended to strongly nonlinear oscillators forced with more than two closely spaced harmonics or even with nonperiodic (transient) excitations.

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