Probabilistic Engineering Mechanics 26 (2011) 119–133

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Effect of stochasticity on targeted energy transfer from a linear medium to a strongly nonlinear attachment

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ARTICLE INFO

Article history:
Received 10 September 2009
Received in revised form
8 November 2010
Accepted 11 November 2010
Available online 24 November 2010

Keywords:
Stochastic targeted energy transfer
Random nonlinear vibrations
Energy harvesting
Fokker–Planck–Kolmogorov equation
Diffusion approximation

ABSTRACT

In this work the problem of targeted energy transfer (TET) from a linear medium to a nonlinear attachment is studied in the presence of stochasticity. Using a Green's function formulation, complexification-averaging technique and diffusion approximation we derive a complex, nonlinear, Ito stochastic differential equation that governs the slow dynamics of the system. Through the numerical solution of the corresponding Fokker–Planck–Kolmogorov (FPK) equation we study the optimal regime of TET and its robustness to stochasticity for the case of nonlinear interactions of the nonlinear attachment with a single mode of the linear system. The probabilistic analysis reveals that in the presence of stochasticity the optimal TET regime, predicted in the deterministic theory, is not only preserved but also is enhanced due to the interaction of nonlinearity and stochasticity.

1. Introduction

In this paper the problem of targeted energy transfer (TET) between a linear structure and a nonlinear attachment is studied in the presence of stochasticity. Targeted energy transfer is a mechanism met in a wide spectrum of both physical phenomena and engineering applications. Examples of physical phenomena connected with TET include resonance-driven solar energy harvesting governing photosynthesis [1] and energy self-focusing, localization and transport governing bioenergetic processes [2]. In engineering applications, energy exchange mechanisms are found in a wide spectrum of scales extending from micro- and nano-applications such as molecular electronic devices [3,4] to large scale structural applications including seismic mitigation problems [5] and energy harvesting from ambient vibrations [6].

Over the past several years, TET has been studied extensively in a deterministic framework. The occurrence of nonlinear TET (or nonlinear energy pumping) was first observed by Gendelman [7], who studied the transient dynamics of a two-DOF system consisting of a damped linear oscillator (designated as the 'primary system') that was weakly coupled to an essentially (strongly) nonlinear, damped attachment; e.g., an oscillator with zero linearized stiffness. A slightly different nonlinear attachment was considered in [8,9]. In these papers the nonlinear oscillator was connected to ground through an essential stiffness nonlinearity. The underlying dynamical mechanism governing TET was found to be a transient resonance capture [10] of the dynamics of the nonlinear attachment on a 1:1 resonance manifold. Nonlinear TET (or nonlinear energy pumping) in two-DOF systems was further investigated in several recent studies. In [11], the onset of nonlinear energy pumping was related to the zero crossing of a frequency of envelope modulation, and a criterion (critical threshold) for inducing nonlinear energy pumping was formulated. A procedure for designing passive nonlinear energy pumping devices was developed in [12] while additional theoretical and numerical results on nonlinear TET were reported in recent works by Gourdon and Lamarque [13] and Gourdon et al. [14]. In [15], the energy exchanges in the damped system were interpreted based on the topological structure and bifurcations of the periodic solutions of the underlying undamped system. Gendelman [16] provided a different perspective of TET dynamics by computing the damped nonlinear normal modes of a linear oscillator coupled to a nonlinear attachment using the invariant manifold approach. Manevitch et al. [17,18], Quinn et al. [19], Koz'min et al. [20] and Sapsis et al. [21] discussed the conditions that should be satisfied by the system and forcing parameters for optimal TET to occur.

In many applications of interest stochasticity or randomness may be involved, which in some cases may drastically alter the deterministic dynamics. For a mathematical model that characterizes a given system, sources of stochasticity may be uncertainties

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doi:10.1016/j.probengmech.2010.11.006

in the initial and boundary conditions, parametric uncertainties including random excitation as well as approximations inherent in the deterministic model. For many of the above cases, deterministic modeling may not be sufficient to give a complete characterization of the system response. On the other hand, a stochastic characterization of the system is more suitable to take into account all the above factors leading to a better understanding of the dynamics under more realistic conditions. Moreover, it provides us with a methodology to evaluate the robustness of the deterministic mechanisms to stochastic perturbations that can model a priori unknown deterministic excitations but with known statistical characteristics; e.g., stochastic spectrum.

Various methods have been developed for the statistical characterization of the response of stochastic systems. These can be categorized based on the nature and dimensionality of the problems that are suitable to handle. For low-dimensional systems a detailed characterization of the system response in terms of the response probability density function (pdf) can be given through a diffusion approximation [22,23] when the stochastic excitation has finite but small correlation time length, or through generalized transport equations governing the joint response-excitation pdf for more general stochastic excitation [24]. Even though the above methods are able to describe complex behaviors of the system under consideration, their applicability is limited to lower dimensional problems due to the serious numerical constraints imposed by the solvability of the corresponding transport equation for the pdf in higher dimensions [25]. For this reason, approximation methods can be applied leading to alternative statistical characterizations from that of the response pdf. Those include closure techniques such as the method of moments [26], cumulant-neglect techniques [27], statistical linearization [28] and particle filter methods [29,30] just to mention a few. For continuous systems the challenges and difficulties are different and hence only very few of the above methodologies can be applied in this context. For this case methods leading to order reduction of the infinite dimensional stochastic space may be applied such as proper orthogonal decomposition [31–33], polynomial-chaos methods [34,35], and dynamically orthogonal field equations [36].

Very few studies have been devoted to the analysis of system responses involving TET in the presence of stochasticity with current results restricted mainly to the numerical study of the mechanism [37,38]. In the present work we consider a general linear system connected with an essentially nonlinear attachment in the presence of stochastic excitation to both the linear system and the nonlinear attachment but also with random initial conditions. Then using a Green's function formulation we transform the full problem to a single, stochastic, nonlinear, intergro-differential equation that captures the full dynamics and contains the complete probabilistic information of the problem. By approximating the dynamics in the early time regime we show that the integral terms may be neglected allowing us to describe the stochastic response though a single stochastic, nonlinear differential equation. The next step of our analysis involves the application of the complexification-averaging technique [39] suitably adapted to treat the stochastic terms of the equation. This leads to a complex, stochastic differential equation governing the slow dynamics of the system in the neighborhood of a resonance capture region. The final step is the approximation of the last equation by an Ito stochastic differential equation through the application of the diffusion or Markov approximation [22,23]. To this end, closed expressions for the correlation time length of the excitation term on the slow dynamics are derived, leading to necessary conditions for the applicability of the Markov approximation. We illustrate the usage of the derived SDE for the slow dynamics in the study of the stochastic nonlinear interactions and TET robustness between a nonlinear attachment and a single mode of a linear structure in the presence of uncertainty.

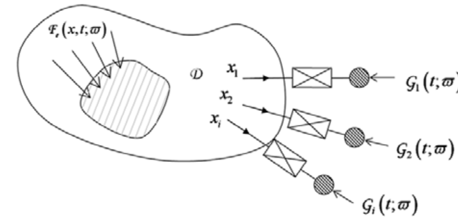

Fig. 1. General linear medium coupled with nonlinear attachments through point connections.

2. Definitions and problem statement

Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a probability space with Ω being the sample space containing the set of elementary events $\varpi \in \Omega$, $\mathcal{B}$ is the σ-algebra associated with Ω, $\mathcal{P}$ is a probability measure, and $t \in T$ denotes time. Then every measurable map of the form $\chi(t;\varpi)$, $\varpi \in \Omega$ is a random function for which we define the mean value operator as

$$\bar{\chi}(t;\varpi) = E^{\varpi}[\chi(t;\varpi)] = \int_{\Omega} \chi(t;\varpi)\, \mathrm{d}\mathcal{P}(\varpi).$$

A zero-mean stochastic process $\chi(t;\varpi), \varpi \in \Omega$ will be called weakly stationary if its correlation function exists for all times and depends on the time difference of the time instants [26,27,22],

$$C_{\chi\chi}(t_1,t_2) \equiv E^{\varpi}[\chi(t_1;\varpi)\,\chi(t_2;\varpi)] = C_{\chi\chi}(\tau), \quad \tau = t_1 - t_2.$$

In this case we can define the correlation time length as [27,22]

$$\tau_c = \frac{\int_0^{\infty} C_{\chi\chi}(\tau)\,\mathrm{d}\tau}{C_{\chi\chi}(0)}$$

which is a measure of 'memory' of the present $\chi(t;\varpi)$ with respect to the past $\chi(s;\varpi)$.

In what follows we will consider a general system that consists of a linear medium coupled with a set of essentially nonlinear oscillators. We assume that the dynamics of the linear medium (which can be discrete or continuous of finite or infinite spatial extent) are governed by a linear operator of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \mathcal{L}[u(x,t)] + \mathcal{F}(x,t;\varpi), \quad x \in \mathcal{D}, \varpi \in \Omega \tag{1}$$

$$\mathcal{M}[u(x,t)] = h(x,t), \quad x \in \partial\mathcal{D}$$

$$u(x,t_0) = f_0(x;\varpi), \qquad u_t(x,t_0) = g_0(x;\varpi), \quad \varpi \in \Omega$$

where x is an index taking values in a discrete or continuous set $\mathcal{D}$ (depending on the nature of the linear medium), $u(x,t)$ is the field describing the response of the linear medium, $\mathcal{M}[u(x,t)]$ is the boundary conditions operator (if applicable), f_0, g_0 describe the initial conditions (possibly stochastic), and $\mathcal{F}$ describes all of the forces acting on the linear medium, deterministic and stochastic.

The linear structure is coupled with a set of essentially nonlinear oscillators connected to the linear medium pointwise at and x_i (Fig. 1).

Following common practice in related literature (see [5], and references therein) we consider the case of cubic nonlinearities where the equation of motion for the attachment has the form

$$m_i\ddot{q}_i + \lambda_i(\dot{q}_i - u_t(x_i,t)) + C_i(q_i - u(x_i,t))^3 = \mathcal{G}_i(t;\varpi) \tag{2}$$

$$q_i(t_0;\varpi) = q_{i0}(\varpi), \qquad \dot{q}_i(t_0;\varpi) = \dot{q}_{i0}(\varpi), \quad \varpi \in \Omega \tag{3}$$

and where m_i, λ_i, C_i, and $\mathcal{G}_i$ are respectively, the mass, damping coefficient, nonlinear stiffness and the external forces acting on the nonlinear attachment. In this case the coupling with the linear medium is taken into account through the relative displacement $(q_i - u(x_i,t))$ across the nonlinear attachment. This particular choice of nonlinear attachment is made since the deterministic dynamics in this case has been studied extensively in the literature (see [5], and references therein) and is well understood. However, the results presented in Sections 3 and 4 can be used for the case of nonlinear stiffness models with general polynomial nonlinearities.

The forces acting on the linear medium can be split into the internal forces due to the nonlinear attachments and the external forces $\mathcal{F}_e(x,t;\varpi)$ as

$$\mathcal{F}(x,t;\varpi) = \sum_i \left[\lambda_i\left(\dot{q}_i - u_t(x_i,t)\right) + C_i\left(q_i - u(x_i,t)\right)^3\right] \times \delta(x - x_i) + \mathcal{F}_e(x,t;\varpi).$$

In the present work we will perform our analysis for the special case of a single attachment coupled to the linear system. The derivations below can be generalized for more attachments although the analytical manipulations may become much more complicated. Hence, for the case of a single nonlinear attachment coupled to the linear system at point x_0, the force acting on the linear system has the form

$$\mathcal{F}(x,t;\varpi) = \left[\lambda\left(\dot{q} - u_t(x_0,t)\right) + C\left(q - u(x_0,t)\right)^3\right] \times \delta(x - x_0) + \mathcal{F}_e(x,t;\varpi). \quad (4)$$

3. Reduction of the dynamics

Based on the linearity of the operators $\mathcal{L}$ and $\mathcal{M}$ we can characterize the response of the linear medium in terms of a resolvent operator, a Green's function that satisfies the problem [40]

$$\frac{\partial^2 G(x,t \mid y,s)}{\partial t^2} = \mathcal{L}\left[G(x,t \mid y,s)\right] + \delta(x-y,t-s) \quad (5a)$$

$$\mathcal{M}\left[G(x,t \mid y,s)\right] = h(x,t), \quad x \in \partial\mathcal{D} \quad (5b)$$

$$G(x,t \mid y,s) = 0, \quad t < s. \quad (5c)$$

By integrating (5a) with respect to time t from s_- to s_+, at fixed x, y and using (5c) to eliminate the lower-limit contributions on the left, we can replace (5c) with more explicit conditions [40]

$$\lim_{t\to s_+} G(x,t \mid y,s) = 0 \qquad \lim_{t\to s_+} G_t(x,t \mid y,s) = \delta(x-y).$$

In this way we can express the response of the linear system as

$$u(x,t) = \int_{t_0}^{t}\int_D G(x,t \mid y,s)\,\mathcal{F}(y,s;\varpi)\,\mathrm{d}y\mathrm{d}s + \int_D \Big[G(x,t \mid y,t_0)\,g_0(y;\varpi) - G_t(x,t \mid y,t_0)\,f_0(y;\varpi)\Big]\mathrm{d}y.$$

Using form (4) for the forces $\mathcal{F}(y,s;\varpi)$ acting on the linear medium, we obtain

$$u(x,t) = \int_{t_0}^{t}\int_D G(x,t \mid y,s)\,\mathcal{F}_e(y,s;\varpi)\,\mathrm{d}y\mathrm{d}s + \int_D \Big[G(x,t \mid y,t_0)\,g_0(y;\varpi) - G_t(x,t \mid y,t_0) \times f_0(y;\varpi)\Big]\mathrm{d}y + \int_{t_0}^{t} G(x,t \mid x_0,s) \times \left[\lambda\left(\dot{q} - u_t(x_0,s)\right) + C\left(q - u(x_0,s)\right)^3\right]\mathrm{d}s. \quad (6)$$

Combining the last expression with the equation of motion for the nonlinear attachment (2), we have the coupled system of equations

$$m\ddot{q} + \lambda\left(\dot{q} - \dot{\zeta}\right) + C\left(q - \zeta\right)^3 = \mathcal{G}(t;\varpi) \quad (7a)$$

$$\zeta(t) = \mathcal{H}(t;\varpi) + \int_{t_0}^{t} G(x_0,t \mid x_0,s) \times \left[\lambda\left(\dot{q} - \dot{\zeta}(s)\right) + C\left(q - \zeta(s)\right)^3\right]\mathrm{d}s \quad (7b)$$

$$q(t_0;\varpi) = q_0(\varpi), \qquad \dot{q}(t_0;(\varpi)) = \dot{q}_0(\varpi), \quad \varpi \in \Omega \quad (7c)$$

where $\zeta(t) \equiv u(x_0,t)$, and the stochastic forcing term $\mathcal{H}(t;\varpi)$ consist of one component due to external random excitation acting on the linear medium and a second component due to random initial conditions. From Eq. (6) it follows that $\mathcal{H}(t;\varpi)$ is given by

$$\mathcal{H}(t;\varpi) = \int_{t_0}^{t}\int_D G(x_0,t \mid y,s)\,\mathcal{F}_e(y,s;\varpi)\,\mathrm{d}y\mathrm{d}s + \int_D \Big[G(x_0,t \mid y,t_0)\,g_0(y;\varpi) - G_t(x_0,t \mid y,t_0)\,f_0(y;\varpi)\Big]\mathrm{d}y.$$

Eq. (4) is a coupled system of a nonlinear ODE and a nonlinear integral equation of the Volterra type of the second kind [41]. Note that system (7) is an exact reformulation of the original problems (1)–(4). Moreover, knowing the response $y(t)$ of the nonlinear attachment and the response of the linear medium at x_0, $\zeta(t)$ we may obtain the response of the whole linear medium through Eq. (6).

Assuming further conditions on the form of the nonlinear attachment as well as on the excitation and initial conditions of the linear medium, we will prove that the coupled response can be approximated by a single stochastic differential equation. Following Vakakis et al. [5] we first apply the transformation

$$\begin{matrix} v = \varepsilon q + \zeta \\ w = q - \zeta \end{matrix} \Leftrightarrow \begin{matrix} q = \dfrac{1}{1+\varepsilon}(v + w) \\ \zeta = \dfrac{1}{1+\varepsilon}(v - \varepsilon w) \end{matrix}$$

to obtain

$$\ddot{w} + \frac{(1+\varepsilon)\lambda}{m}\dot{w} + \frac{(1+\varepsilon)C}{m}w^3 = \frac{(1+\varepsilon)}{m}\mathcal{G}(t;\varpi) - \ddot{v}$$

$$\frac{1}{1+\varepsilon}\left(v(t) - \varepsilon w(t)\right) = \mathcal{H}(t;\varpi) + \int_{t_0}^{t} G(x_0,t \mid x_0,s)\left[\lambda\dot{w}(s) + C\left(w(s)\right)^3\right]\mathrm{d}s$$

$$w(t_0;\varpi) = q_0(\varpi), \qquad \dot{w}(t_0;\varpi) = -g_0(x_0;\varpi) + \dot{q}_0(\varpi)$$

where we have assumed without loss of generality $f_0(x_0;\varpi) = 0$. Substituting the value $v(t)$ from the second equation into the first equation, we obtain

$$\ddot{w} + \frac{\lambda}{m}\dot{w} + \frac{C}{m}w^3 = \frac{1}{m}\mathcal{G}(t;\varpi) - \ddot{\mathcal{H}}(t;\varpi) - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_{t_0}^{t} G(x_0,t \mid x_0,s)\left[\lambda\dot{w}(s) + C\left(w(s)\right)^3\right]\mathrm{d}s\right]. \quad (8)$$

The last stochastic, nonlinear integrodifferential equation is an exact reformulation of the original problem, and no approximations have been made. However, in order to proceed with our analysis, we will now assume that the nonlinear attachment has small mass and is weakly damped, and that the initial conditions and external force acting on the linear medium are also small. More specifically, we assume that

$$m \to \varepsilon m, \qquad \lambda \to \varepsilon\lambda, \qquad \mathcal{G} \to \varepsilon^{\frac{3}{2}}\mathcal{G}, \qquad \mathcal{F}_e \to \varepsilon^{\frac{1}{2}}\mathcal{F}_e$$

$$f_0 \to \varepsilon^{\frac{1}{2}}f_0, \qquad g_0 \to \varepsilon^{\frac{1}{2}}g_0, \qquad q_0 \to \varepsilon^{\frac{1}{2}}q_0, \qquad \dot{q}_0 \to \varepsilon^{\frac{1}{2}}\dot{q}_0$$

which also induce the rescalings

$$w \to \varepsilon^{\frac{1}{2}}w, \qquad \mathcal{H} \to \varepsilon^{\frac{1}{2}}\mathcal{H}$$

where $\varepsilon \ll 1$ is the scaling parameter. Applying the above rescalings we obtain

$$\ddot{w}+\frac{\lambda}{m}\dot{w}+\frac{C}{m}w^{3} =\frac{1}{m}\mathcal{G}(t;\varpi)-\ddot{\mathcal{H}}(t;\varpi) -\varepsilon\frac{\mathrm{d}^{2}}{\mathrm{d}t^{2}}\left[\int_{t_0}^{t}G(x_0,t\mid x_0,s)\left[\lambda\dot{w}(s)+C(w(s))^{3}\right]\mathrm{d}s\right]. \quad (9)$$

The first term on the right hand side expresses the external forces acting directly on the nonlinear attachment. The second term is the forces acting indirectly on the nonlinear attachment due to the dynamics of the linear medium created by the initial conditions and the external forces acting on it. Finally, the last term expresses the force due to the modified dynamics of the linear medium due to the presence of the nonlinear attachment. Its effect becomes important in large time scales of order ε^{-1} and, therefore, for the short time regime the dynamics can be approximated by the nonlinear equation

$$\ddot{w}+\hat{\lambda}\dot{w}+\hat{C}w^{3}=\widehat{\mathcal{G}}(t;\varpi)-\ddot{\mathcal{H}}(t;\varpi)+\mathcal{O}(\varepsilon) \quad (10a)$$

$$w(t_0;\varpi)=q_0(\varpi),\quad \dot{w}(t_0;\varpi)=-g_0(x_0;\varpi)+\dot{q}_0(\varpi) \quad (10b)$$

where $\hat{\lambda}=\lambda/m$, $\hat{C}=C/m$, and $\widehat{\mathcal{G}}=\frac{1}{m}\mathcal{G}$.

4. Equation governing the stochastic slow dynamics

We will now assume that the external forces acting directly on the nonlinear attachment can be expressed as a zero-mean stationary stochastic process; e.g., white noise. We also assume that the forces acting on the linear medium can be decomposed into a deterministic part and a zero-mean, stationary, stochastic part so that

$$\mathcal{H}(t;\varpi)=\mathcal{H}_d(t)+\mathcal{H}_s(t;\varpi)+\mathcal{H}_0(t;\varpi)$$

where $\mathcal{H}_d(t)$ is the deterministic component, $\mathcal{H}_s(t;\varpi)$ is the zero-mean stochastic component due to random forces acting on the linear medium, and $\mathcal{H}_0(t;\varpi)$ is a zero-mean stochastic component due to random initial conditions of the linear medium. Each of the above terms is analyzed below.

4.1. Deterministic part $\mathcal{H}_d(t)$

By considering the mean of $\mathcal{H}(t;\varpi)$ we have for the deterministic part

$$\mathcal{H}_d(t)=\int_{t_0}^{t}\int_D G(x_0,t\mid y,s)\bar{\mathcal{F}}_e(y,s)\,\mathrm{d}y\mathrm{d}s \quad (11)$$

where $\bar{\mathcal{F}}_e$, is the deterministic (mean) part of the forcing acting on the linear medium; i.e., $E^{\varpi}[\mathcal{F}_e(y,s;\varpi)]$.

4.2. Stochastic part due to random forces $\mathcal{H}_s(t;\varpi)$

The term $\mathcal{H}_s(t;\varpi)$ is the zero-mean stochastic component due to the randomness of the external forces and is given by

$$\mathcal{H}_s(t;\varpi)=\int_{t_0}^{t}\int_D G(x_0,t\mid y,s)\mathcal{F}'(y,s;\varpi)\,\mathrm{d}y\mathrm{d}s$$

with $\mathcal{F}'(y,s;\varpi)=\mathcal{F}_e(y,s;\varpi)-\bar{\mathcal{F}}_e(y,s;\varpi)$ being the zero-mean stochastic fluctuation of the forces acting on the linear medium, in general non-stationary. In this case its correlation function will be given by

$$C_{\mathcal{H}_s\mathcal{H}_s}(t_1,t_2)=\int_{t_0}^{t_1}\int_{t_0}^{t_2}\int_D\int_D G(x_0,t_1\mid y_1,s_1)G(x_0,t_2\mid y_2,s_2)\times C_{\mathcal{F}'\mathcal{F}'}(s_1,s_2,y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2$$

where

$$C_{\mathcal{F}'\mathcal{F}'}(t_1,t_2,y_1,y_2)=E^{\varpi}\left[\mathcal{F}'(y_1,s_1;\varpi)\mathcal{F}'(y_2,s_2;\varpi)\right].$$

Assuming a spatially decorrelation field $\mathcal{F}'(y,s;\varpi)$ gives

$$C_{\mathcal{F}'\mathcal{F}'}(t_1,t_2,y_1,y_2)=\tilde{C}_{\mathcal{F}'\mathcal{F}'}(t_1,t_2,y_1)\,\delta(y_1-y_2).$$

This is the case where the forces acting on two different points of the linear medium are statistically independent. Then, the expression for the correlation function is given by

$$C_{\mathcal{H}_s\mathcal{H}_s}(t_1,t_2)=\int_{t_0}^{t_1}\int_{t_0}^{t_2}\int_D G(x_0,t_1\mid y,s_1)G(x_0,t_2\mid y,s_2)\times\tilde{C}_{\mathcal{F}'\mathcal{F}'}(s_1,s_2,y)\,\mathrm{d}y\mathrm{d}s_1\mathrm{d}s_2. \quad (12)$$

In what follows we will assume for simplicity that $\mathcal{H}_s(t;\varpi)$ is a stationary stochastic process even though the analysis can be done for the general case. The above assumption can be made for the case of a linear time invariant medium where $G(x,t\mid y,s)=G(x,t-s\mid y)$, and the stochastic process $\mathcal{F}'(y,s;\varpi)$ is stationary.

4.3. Excitation term due to initial conditions $\mathcal{H}_0(t;\varpi)$

Finally, the term $\mathcal{H}_0(t;\varpi)$ reflects the effect of the stochastic part of the initial conditions for the linear medium and is given by

$$\mathcal{H}_0(t;\varpi)=\int_D\Big[G(x_0,t\mid y,t_0)g_0(y;\varpi)-G_t(x_0,t\mid y,t_0)f_0(y;\varpi)\Big]\mathrm{d}y.$$

4.4. Stochastic averaging

We now write Eq. (10a) in the form

$$\ddot{w}+\hat{\lambda}\dot{w}+\hat{C}w^{3}=\Phi_d(t)+\Phi_s(t;\varpi)+\mathcal{O}(\varepsilon) \quad (13)$$

where $\Phi_d(t)=\ddot{\mathcal{H}}_d(t)-\ddot{\mathcal{H}}_0(t;\varpi)$, and $\Phi_s(t;\varpi)=\widehat{\mathcal{G}}(t;\varpi)+\ddot{\mathcal{H}}_s(t;\varpi)$ is a zero-mean stochastic process with known statistical characteristics defined by the forcing terms and the initial conditions.

Clearly the linear medium may have many natural frequencies. As is demonstrated in [5] the nonlinear attachment engages subsequently with each of the linear modes and dissipates energy through a resonance capture cascade. Here, we want to study the interaction of the nonlinear attachment with a single frequency ω_0 of the linear system under the condition of 1:1 transient resonance capture. Therefore, during this transient resonance capture, we consider only the part of the spectrum of the linear system that is close to the considered frequency ω_0 and which can be expressed as

$$\Phi_d(t)=\mathrm{e}^{j\omega_0 t}\int_{-\delta}^{\delta}S_{\Phi_d}(\omega+\omega_0)\,\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

where S_{Φ_d} is the Fourier transform of the function $\Phi_d(t)$, $S_{\Phi_d}(\omega)=\frac{1}{2\pi}\int\mathrm{e}^{-j\omega t}\Phi_d(t)\mathrm{d}t$, and δ is a small parameter relative to ε which allows us to take into account closely spaced modes that can play a significant role in the dynamics of the attachment during the 1:1 transient resonance capture. Now, the goal is to derive a modulation equation governing the stochastic slow dynamics of the system during this phase. To this end we will adapt the complexfication-averaging method developed by Manevitch [39] to the present case where uncertainty has to be taken into account. We introduce the new complex variable

$$\psi(t;\varpi)=\dot{w}(t;\varpi)+jw(t;\varpi).$$

Substituting, Eq. (13) is expressed as

$$\dot{\psi} - \frac{j}{2}\left(\psi + \psi^*\right) + \frac{\hat{\lambda}}{2}\left(\psi + \psi^*\right) + \frac{j\hat{C}}{8}\left(\psi - \psi^*\right)^3 = \mathrm{e}^{j\omega_0 t}\int_{-\delta}^{\delta} S_{\Phi_d}\left(\omega + \omega_0\right)\mathrm{e}^{j\omega t}\mathrm{d}\omega + \xi\left(t;\varpi\right).$$

We represent the solution of the equation for ψ as

$$\psi\left(t;\varpi\right) = z\left(t;\varpi\right)\mathrm{e}^{j\omega_0 t}$$

where $z\left(t;\varpi\right)$ is a slowly varying complex amplitude that modulates the fast oscillation $\mathrm{e}^{j\omega_0 t}$. Subsequently, we average out the fast frequency to obtain

$$\dot{z} + \frac{j}{2}z + \frac{\hat{\lambda}}{2}z - \frac{3j\hat{C}}{8}|z|^2 z = \int_{-\delta}^{\delta} S_{\Phi_d}\left(\omega + \omega_0\right)\mathrm{e}^{j\omega t}\mathrm{d}\omega + \eta\left(t;\varpi\right) \tag{14}$$

$$z\left(t_0\right) = \mathrm{e}^{-j\omega_0 t_0}\left(\dot{w}\left(t_0\right) + jw\left(t_0\right)\right) \tag{15}$$

where, as noted earlier, the spectral density S_{Φ_d} is narrowly distributed around ω_0; therefore, the averaging of this term is justified. The last term is a stochastic process given by

$$\eta\left(t;\varpi\right) = \frac{\omega_0}{2\pi}\int_{t-\frac{\pi}{\omega_0}}^{t+\frac{\pi}{\omega_0}} \mathrm{e}^{-j\omega_0 s}\Phi_s\left(s;\varpi\right)\mathrm{d}s. \tag{16}$$

Clearly, the above process is zero-mean and, moreover, its correlation function is given by

$$C_{\eta\bar{\eta}}\left(t_1,t_2\right) = \left(\frac{\omega_0}{2\pi}\right)^2\int_{t_1-\frac{\pi}{\omega_0}}^{t_1+\frac{\pi}{\omega_0}}\int_{t_2-\frac{\pi}{\omega_0}}^{t_2+\frac{\pi}{\omega_0}} \mathrm{e}^{-j\omega_0(s_1-s_2)} \times C_{\Phi_s\Phi_s}\left(s_1 - s_2\right)\mathrm{d}s_1\mathrm{d}s_2.$$

By applying the transformation $s_i \to s_i + t_i$ we conclude that the process is stationary and, by setting $\tau = t_1 - t_2$, we obtain

$$\begin{aligned} C_{\eta\eta^*}\left(\tau\right) &= \left(\frac{\omega_0}{2\pi}\right)^2\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \mathrm{e}^{-j\omega_0(\tau+s_1-s_2)} \times C_{\Phi_s\Phi_s}\left(\tau + s_1 - s_2\right)\mathrm{d}s_1\mathrm{d}s_2 \\ &= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\left[\mathrm{e}^{-j\omega_0(\tau-s)}C_{\Phi_s\Phi_s}\left(\tau - s\right) + \mathrm{e}^{-j\omega_0(\tau+s)}C_{\Phi_s\Phi_s}\left(\tau + s\right)\right]\left(1 - \frac{\omega_0 s}{2\pi}\right)\mathrm{d}s. \end{aligned} \tag{17}$$

Hence, we have derived the stochastic differential equation (14) that governs the motion of the slow dynamics. Eq. (14) depends on S_{Φ_d}, the Fourier transform of $\Phi_d(t)$, and on the zero-mean stochastic process $\eta\left(t;\varpi\right)$ that has a correlation function given by (17).

4.5. Correlation time length for $\eta(t;\varpi)$

We shall now derive a transport equation governing the evolution of the probability density function $\chi\left(z,t\right)$ describing the stochastic, slowly varying, complex amplitude $z\left(t;\varpi\right)$. To simplify our analysis we assume that the initial conditions that characterize the linear medium are deterministic so that $S_{\Phi_d}\left(\omega\right)$ is also a deterministic quantity.

To derive a transport equation for $\chi\left(z,t\right)$ we will apply the diffusion approximation [22,23] to the stochastic differential equation (14). To this end we need to estimate the correlation length of the stochastic process $\eta\left(t;\varpi\right)$. We have

$$\begin{aligned} \int_0^\infty C_{\eta\eta^*}\left(\tau\right)\mathrm{d}\tau &\le \int_0^\infty \left|C_{\eta\eta^*}\left(\tau\right)\right|\mathrm{d}\tau \\ &\le \frac{\omega_0}{2\pi}\int_0^\infty\int_0^{\frac{2\pi}{\omega_0}}\left[\left|C_{\Phi_s\Phi_s}\left(\tau - s\right)\right| + \left|C_{\Phi_s\Phi_s}\left(\tau + s\right)\right|\right]\left(1 - \frac{\omega_0 s}{2\pi}\right)\mathrm{d}s\mathrm{d}\tau \\ &= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\left(1 - \frac{\omega_0 s}{2\pi}\right)\int_0^\infty\left[\left|C_{\Phi_s\Phi_s}\left(\tau - s\right)\right| + \left|C_{\Phi_s\Phi_s}\left(\tau + s\right)\right|\right]\mathrm{d}\tau\mathrm{d}s \\ &= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\left(1 - \frac{\omega_0 s}{2\pi}\right)\left(\int_{-s}^\infty\left|C_{\Phi_s\Phi_s}\left(\tau\right)\right|\mathrm{d}\tau + \int_s^\infty\left|C_{\Phi_s\Phi_s}\left(\tau\right)\right|\mathrm{d}\tau\right)\mathrm{d}s \\ &= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\left(1 - \frac{\omega_0 s}{2\pi}\right)\mathrm{d}s\int_{-\infty}^\infty\left|C_{\Phi_s\Phi_s}\left(\tau\right)\right|\mathrm{d}\tau \\ &= \int_0^\infty\left|C_{\Phi_s\Phi_s}\left(\tau\right)\right|\mathrm{d}\tau. \end{aligned}$$

Moreover,

$$C_{\eta\eta^*}(0) = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}}\cos\left(\omega_0 s\right)C_{\Phi_s\Phi_s}(s)\left(1 - \frac{\omega_0 s}{2\pi}\right)\mathrm{d}s.$$

Therefore we have an upper bound for the correlation time length

$$\tau_\eta \le \frac{\frac{\pi}{\omega_0}\int_0^\infty\left|C_{\Phi_s\Phi_s}\left(\tau\right)\right|\mathrm{d}\tau}{\int_0^{\frac{2\pi}{\omega_0}}C_{\Phi_s\Phi_s}(s)\left(1 - \frac{\omega_0 s}{2\pi}\right)\cos\left(\omega_0 s\right)\mathrm{d}s} = \frac{\tau_{\Phi_s}}{2}\left[\int_0^1\frac{C_{\Phi_s\Phi_s}\left(\frac{s\pi}{\omega_-}\right)}{C_{\Phi_s\Phi_s}(0)}\cos\left(\pi s\right)\left(1 - s\right)\mathrm{d}s\right]^{-1}.$$

The first form is more suitable for fast decaying correlation functions with respect to the time scale $\frac{\pi}{\omega_0}$. For this case, in the limit of very fast decaying $C_{\Phi_s\Phi_s}(s)$ we will have

$$\int_0^{\frac{2\pi}{\omega_0}}C_{\Phi_s\Phi_s}(s)\left(1 - \frac{\omega_0 s}{\pi}\right)\cos\left(\omega_0 s\right)\mathrm{d}s \simeq \int_0^{\frac{2\pi}{\omega_0}}C_{\Phi_s\Phi_s}(s)\mathrm{d}s = \int_0^\infty C_{\Phi_s\Phi_s}(s)\mathrm{d}s \simeq \int_0^\infty\left|C_{\Phi_s\Phi_s}(s)\right|\mathrm{d}s$$

and therefore

$$\tau_\eta \le \frac{\pi}{\omega_0}.$$

Such a special case is when $\Phi_s\left(t;\varpi\right)$ is white noise where we have $C_{\Phi_s\Phi_s}\left(\tau\right) = \nu^2\delta\left(\tau\right)$ (which is the correlation function with the fastest possible decaying or with the smallest memory). Then we can calculate directly from (17) that

$$\tau_{\eta,w} = \frac{\pi}{2\omega_0}. \tag{18}$$

This is the smallest possible correlation length that the stochastic process $\eta\left(t;\varpi\right)$ may have and is dominated not from the correlation length of Φ_s (which is zero) but rather from the averaging procedure that is over a time scale $T = \frac{\pi}{\omega_0}$ (averaging/integration is a process that introduces memory to a stochastic process).

On the other hand if $C_{\Phi_s\Phi_s}$ is sufficiently slowly decaying with respect to the time scale $\frac{\pi}{\omega_0}$ then the second form of the upper bound is more suitable. Using the latter we obtain

$$\tau_\eta < \frac{\tau_{\Phi_s}}{2}\left[\int_0^1 \frac{C_{\Phi_s\Phi_s}\left(\frac{s\pi}{\omega_-}\right)}{C_{\Phi_s\Phi_s}(0)}\cos(\pi s)(1-s)\,\mathrm{d}s\right]^{-1} \simeq \frac{\tau_{\Phi_s}}{2}\left[\int_0^1 \cos(\pi s)(1-s)\,\mathrm{d}s\right]^{-1} = \frac{\pi^2}{4}\tau_{\Phi_s}$$

and therefore in this case the primary role is played by the correlation length of the stochastic process Φ_s.

The next step is to derive an estimate for the correlation time length τ_{Φ_s}. Since $\mathcal{F}'(y,s;\varpi)$ and $\widehat{\mathcal{G}}(t;\varpi)$ are zero-mean stochastic fluctuations acting on the linear medium and the nonlinear attachment, respectively, we assume that they have characteristics of white noise. Then, if the intensity of $\widehat{\mathcal{G}}(t;\varpi)$ is non-zero, the correlation time length of the process $\Phi_s(t;\varpi) = \widehat{\mathcal{G}}(t;\varpi) + \ddot{\mathcal{H}}_s(t;\varpi)$ is independent of the stochastic characteristics of $\ddot{\mathcal{H}}_s(t;\varpi)$ and is equal to zero.

However, for the case where $\widehat{\mathcal{G}}(t;\varpi) = 0$, the correlation time length of $\Phi_s(t;\varpi)$ is governed completely by the stochastic characteristics of the process $\mathcal{F}'(y,s;\varpi)$ and the dynamics of the linear medium described by the Green's function G. To proceed with our analysis we consider a special form for the correlation function for the stochastic fluctuation $\mathcal{F}'(y,s;\varpi)$ given by

$$C_{\mathcal{F}'\mathcal{F}'}(t_1,t_2,y_1,y_2) = \nu^2_{\mathcal{F}'}(y_1,y_2)\,\delta(t_1-t_2).$$

In this case we will have from Eq. (12)

$$\begin{aligned} C_{\ddot{\mathcal{H}}_s\ddot{\mathcal{H}}_s}(t+\tau,t) &= \int_{t_0}^{t+\tau}\int_{t_0}^{t}\int_D\int_D G_{tt}(x_0,t+\tau-s_1\mid y_1) \\ &\quad\times G_{tt}(x_0,t-s_2\mid y_2)\,\nu^2_{\mathcal{F}'}(y_1,y_2)\,\delta(s_1-s_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2 \\ &= \int_{t_0}^{t}\int_{t_0}^{t}\int_D\int_D G_{tt}(x_0,t+\tau-s_1\mid y_1)\,G_{tt}(x_0,t-s_2\mid y_2) \\ &\quad\times \nu^2_{\mathcal{F}'}(y_1,y_2)\,\delta(s_1-s_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2 \\ &= \int_{t_0}^{t}\int_D\int_D G_{tt}(x_0,t+\tau-s\mid y_1)\,G_{tt}(x_0,t-s\mid y_2) \\ &\quad\times \nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s. \end{aligned}$$

Therefore,

$$C_{\ddot{\mathcal{H}}_s\ddot{\mathcal{H}}_s}(\tau,\delta t) = \int_0^{\delta t}\int_D\int_D G_{tt}(x_0,\tau+s\mid y_1)\,G_{tt}(x_0,s\mid y_2) \times \nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s, \quad \delta t = t-t_0.$$

From this it follows that

$$C_{\ddot{\mathcal{H}}_s\ddot{\mathcal{H}}_s}(0,\delta t) = \int_0^{\delta t}\int_D\int_D G_{tt}(x_0,s\mid y_1)\,G_{tt}(x_0,s\mid y_2) \times \nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s$$

and

$$\begin{aligned} \int_0^\infty C_{\ddot{\mathcal{H}}_s\ddot{\mathcal{H}}_s}(\tau,\delta t)\,\mathrm{d}\tau &= \int_0^{\delta t}\int_D\int_D\left(\int_0^\infty G_{tt}(x_0,\tau+s\mid y_1)\,\mathrm{d}\tau\right) \\ &\quad\times G_{tt}(x_0,s\mid y_2)\,\nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s \\ &= -\int_0^{\delta t}\int_D\int_D G_t(x_0,s\mid y_1)\,G_{tt}(x_0,s\mid y_2) \\ &\quad\times \nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s. \end{aligned}$$

Then, the correlation length τ_{Φ_s} can be characterized by the functional $\mathcal{T}\left[G,\nu^2_{\mathcal{F}'};\delta t\right]$ defined as

$$\tau_{\Phi_s}(\delta t) = -\frac{\int_0^{\delta t}\int_D\int_D G_t(x_0,s\mid y_1)\,G_{tt}(x_0,s\mid y_2)\,\nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s}{\int_0^{\delta t}\int_D\int_D G_{tt}(x_0,s\mid y_1)\,G_{tt}(x_0,s\mid y_2)\,\nu^2_{\mathcal{F}'}(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s} \equiv \mathcal{T}\left[G,\nu^2_{\mathcal{F}'}\right]$$

Thus, we have an expression for the correlation length in terms of the Green's function that characterizes the linear medium, and now have upper bounds and approximations for the correlation time length characterizing the stochastic process $\eta(t;\varpi)$ for various cases. Using those we will prove that the diffusion approximation can be applied to system (14) in order to derive a transport equation for the probability density function characterizing the slow dynamics of the system.

4.6. Diffusion approximation

The diffusion method relies on the approximation of the stochastic process $\eta(t;\varpi)$ by a process with independent increments. The process $\eta(t;\varpi)$ does not possess the property of independent increments since, as we proved, it has finite correlation time length. However, if the deterministic dynamics governing the evolution of the stochastic process acts on a sufficiently slower time scale than the memory of the stochastic process, then the independent increment approximation is valid [22,23].

As we showed in the previous section, when the intensity of the noise $\widehat{\mathcal{G}}(t;\varpi)$ is non-zero, then $\tau_{\Phi_s} = 0$ and, therefore, $\tau_\eta \le \frac{\pi}{\omega_0}$. Hence, the memory of the stochastic process $\eta(t;\varpi)$ is sufficiently smaller than the slow dynamics described by the slow flow Eq. (14) in the absence of noise, and the diffusion approximation is valid.

For the case where $\widehat{\mathcal{G}}(t;\varpi) = 0$, the correlation time length is bounded by

$$\tau_\eta < \frac{\pi^2}{4}\tau_{\Phi_s}(\delta t). \tag{19}$$

Therefore, a sufficient condition for the validity of the diffusion approximation is $\frac{\pi^2}{4}\tau_{\Phi_s}(\delta t) = \mathcal{O}\left(\frac{\pi}{\omega_0}\right)$.

Under these assumptions, Eq. (14) can be approximated by the Ito stochastic differential equation

$$\mathrm{d}z = \left[-\frac{j}{2}z - \frac{\hat{\lambda}}{2}z + \frac{3j\hat{C}}{8}|z|^2 z + \int_{-\delta}^{\delta} S_{\Phi_d}(\omega+\omega_0)\,\mathrm{e}^{j\omega t}\mathrm{d}\omega\right] \times \mathrm{d}t + \sigma_i\mathrm{d}W_i(t;\varpi) \tag{20}$$

where $\sigma_i = \sigma_{1i} + j\sigma_{2i}$ is such that

$$\sigma_{ik}\sigma_{kj} = \frac{1}{\tau_\eta}\int_{-\frac{\tau_\eta}{2}}^{\frac{\tau_\eta}{2}}\int_{-\frac{\tau_\eta}{2}}^{\frac{\tau_\eta}{2}} C_{\eta_i\eta_j}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 \tag{21}$$

with $\eta(t_1;\varpi) = \eta_1(t_1;\varpi) + j\eta_2(t_1;\varpi)$.

Note that, for the case where the uncertainty is introduced only through the forcing directly on the attachment ($\mathcal{F}'(y,s;\varpi) = 0$) which has white noise characteristics $C_{\Phi_s\Phi_s}(t,s) = \nu^2\delta(t-s)$, we can use Eq. (16) to obtain

$$\sigma_{ik}\sigma_{kj} = \frac{1}{\tau_\eta}\int_{-\frac{\tau_\eta}{2}}^{\frac{\tau_\eta}{2}}\int_{-\frac{\tau_\eta}{2}}^{\frac{\tau_\eta}{2}} C_{\eta_i\eta_j}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 = \tau_{\eta,w}\frac{\nu^2\omega_0}{4\pi}\delta_{ij} = \frac{\nu^2}{8}\delta_{ij} \tag{22}$$

where in the last equation we have used the correlation time length that we computed in (18).

Subsequently, we express the complex amplitude as real and imaginary components $z = z_1 + jz_2$. Then, Eq. (14) can be expressed as

$$\mathrm{d}z_1 = \Xi_1(z_1, z_2, t)\,\mathrm{d}t + \sigma_{i1}\mathrm{d}W_i(t; \varpi)$$
$$\mathrm{d}z_2 = \Xi_2(z_1, z_2, t)\,\mathrm{d}t + \sigma_{i2}\mathrm{d}W_i(t; \varpi)$$

where

$$\Xi_1(z_1, z_2, t) = \frac{1}{2}z_2 - \frac{\hat{\lambda}}{2}z_1 - \frac{3\hat{C}}{8}\left(z_1^2 + z_2^2\right)z_2 + \mathrm{Re}\int S_{\Phi_d}(\omega + \omega_0)\,\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

$$\Xi_2(z_1, z_2, t) = -\frac{1}{2}z_1 - \frac{\hat{\lambda}}{2}z_2 + \frac{3\hat{C}}{8}\left(z_1^2 + z_2^2\right)z_1 + \mathrm{Im}\int S_{\Phi_d}(\omega + \omega_0)\,\mathrm{e}^{j\omega t}\mathrm{d}\omega.$$

Moreover, the last SDE is equivalent to the following Fokker–Planck–Kolmogorov (FPK) equation for the probability density function $\chi(z_1, z_2, t)$

$$\frac{\partial \chi}{\partial t} + \frac{\partial(\Xi_1(z_1, z_2, t)\,\chi)}{\partial z_1} + \frac{\partial(\Xi_2(z_1, z_2, t)\,\chi)}{\partial z_2} - \frac{1}{2}\sigma_{ik}\sigma_{kj}\frac{\partial^2 \chi}{\partial z_i \partial z_j} = 0. \tag{23}$$

5. Interaction with a single linear mode

We will now apply the above analysis to study the interaction of the nonlinear attachment with a single mode of the linear system under the condition of 1:1 transient resonance capture and in the presence of noise having the same order of magnitude as the response of the system. The deterministic dynamics of this system have been studied previously [19,21,5] in the context of TET.

Here we wish to study these dynamics in the presence of stochasticity as well as the robustness of the TET mechanisms observed in the deterministic context under the influence of stochastic perturbations. We will assume that no external forces act on the system except for a zero-mean stochastic perturbation with white noise characteristics which acts directly on the nonlinear attachment. The last assumption doesn't introduce any restrictions from treating the general case of stochastic excitation; the only change is the calculation of the diffusion coefficient. In what follows the diffusion coefficient will be chosen to be of the same order as the system response near the critical TET regime so that we can get a characterization of the robustness properties of the system when randomness magnitude is comparable with the system response. Moreover, we will consider the system to be excited by deterministic initial conditions having the form

$$f_0(y; \varpi) = 0 \quad \text{and} \quad g_0(y; \varpi) = g_0(y)$$
$$q_0 = \dot{q}_0 = 0.$$

Then,

$$\Phi_d(t) = -\ddot{\mathcal{H}}_0(t; \varpi) = -\int_D G_{tt}(x_0, t \mid y, t_0)\, g_0(y)\mathrm{d}y.$$

Since our study will focus on the interaction of the nonlinear attachment with a single mode of the linear medium (under the condition of 1:1 transient resonance capture), we consider a Green's function that has the form

$$G(x, t \mid y, s) = \sum_i A_i(x \mid y)\sin(\omega_i[t - s])$$

where the frequencies are well separated from each other (with respect to the scale ε). Assuming interaction with a single mode (at frequency ω_0) we obtain

$$\int_{-\delta}^{\delta} S_{\Phi_d}(\omega + \omega_0)\,\mathrm{e}^{j\omega t}\mathrm{d}\omega = -\frac{j}{2}\omega_0^2\int_D A_0(x_0 \mid y)\, g_0(y)\mathrm{d}y.$$

Then, Eq. (20) takes the form

$$\mathrm{d}z = \left[-\frac{j}{2}z - \frac{\hat{\lambda}}{2}z + \frac{3j\hat{C}}{8}|z|^2 z - \frac{j}{2}B_1\right]\mathrm{d}t + \sigma_i \mathrm{d}W_i(t; \varpi) \tag{24}$$

$$z(0) = -B_2 \tag{25}$$

where $B_1 = \omega_0^2\int_D A_0(x_0 \mid y)\, g_0(y)\mathrm{d}y$ and $B_2 = g_0(x_0)$ depend on the initial conditions of the linear medium at $t_0 = 0$. The last equation approximates the dynamics up to times of $\mathcal{O}(\varepsilon^{-1})$. Moreover, since we assumed that the only external force is the white noise acting on the nonlinear oscillator, the diffusion coefficient for the FPK equation is given by (22). As mentioned earlier, the assumption about the form of stochasticity doesn't introduce restrictions on the analysis since the general case of stochastic excitation acting on both the linear system and the nonlinear attachment will also lead to the slow flow Eq. (24) having a constant diffusion coefficient given by Eq. (21) and with the assumption that condition (19) for a small correlation time length is satisfied.

An important measure that we will use in the sequel to evaluate the performance of TET is the total energy dissipated by the nonlinear oscillator. As shown in [21] this is given by (adapted for the stochastic case)

$$E_d(t) = \frac{\varepsilon\hat{\lambda}}{2}\int_0^t E^{\varpi}\left[|z(t; \varpi)|^2\right]\mathrm{d}t. \tag{26}$$

Therefore, we have strong dissipation of energy by the nonlinear attachment when the modulus of the slow-flow, $|z(t; \varpi)|$, is larger during the initial stage of motion. In what follows we will assume that $B_1 = B_2 \equiv B$, a case that occurs when the linear medium is a single degree of freedom oscillator with natural frequency $\omega_0 = 1$.

5.1. An overview of the deterministic dynamics

As shown in [19,21], the deterministic dynamics of (24) are strongly influenced by the underlying topological structure of the periodic and quasi-periodic orbits of the Hamiltonian system. More specifically, the undamped, deterministic system is integrable with Hamiltonian

$$\frac{j}{2}|z|^2 - \frac{j}{4}|z|^4 - \frac{jB}{2}z^* + \frac{jB}{2}z = h.$$

Taking into account the integrability of the system we can reduce it to a one-dimensional slow-flow

$$2\dot{a} = [f(a; B)]^{1/2}, \quad f(a; B) = 4Ba - \left(a - \frac{a^2}{2} + \frac{B^4}{2} + B^2\right)^2$$

where $a(t) = |z(t)|^2$. Depending on the value of $B, f(a; B)$ can possess two, three or four real roots. For a critical value $B = B_{cr} = 0.3672$, two of the real roots coincide (case of three real roots) and, as shown in [21] for this case the system possesses a homoclinic orbit that is determined explicitly in terms of the two homoclinic loops (Fig. 2)

$$|z(t)|^2 = a_h^{(-)}(t) = a_2 - \frac{\gamma_1\gamma_2}{\gamma_1\sinh^2\left(\frac{\sqrt{\gamma_1\gamma_2}}{8}t\right) + \gamma_2\cosh^2\left(\frac{\sqrt{\gamma_1\gamma_2}}{8}t\right)} \tag{27}$$

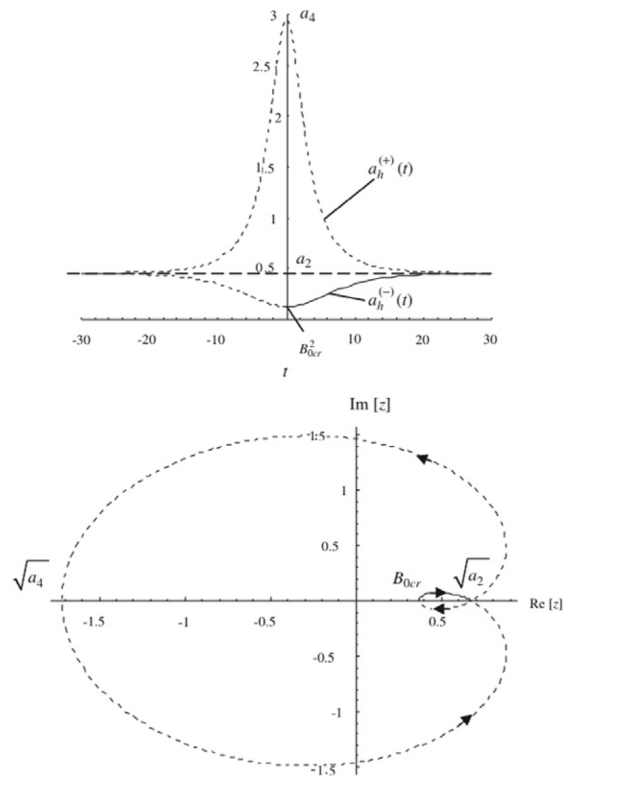

Fig. 2. Homoclinic orbit in terms of $|z(t)|^2$ (upper plot) and also shown in the slow-flow phase space: Re[z], Im[z] (lower plot).

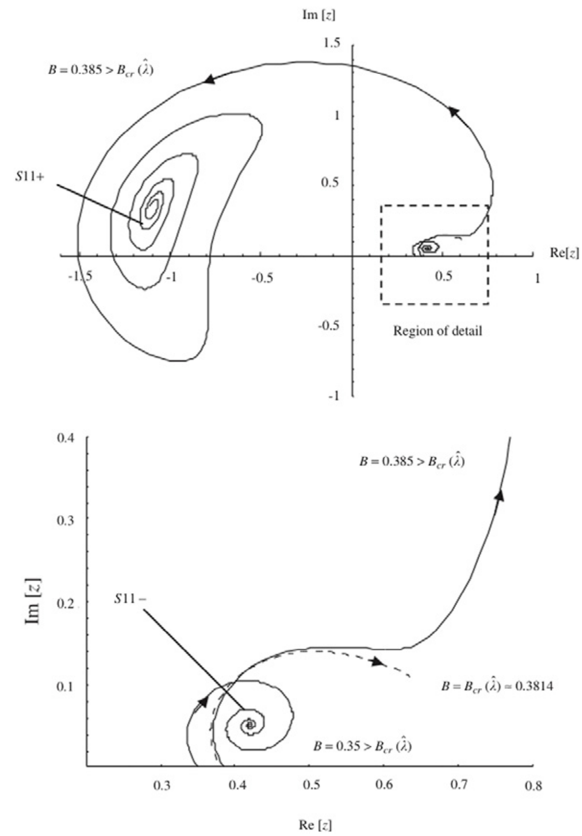

Fig. 3. Damped transition of the system initiated close to the perturbed homoclinic orbit defined by $B_{\mathrm{cr}}\left(\hat{\lambda}\right) \simeq 0.3814$.

and

$$|z(t)|^2 = a_h^{(+)}(t) = a_2 + \frac{\gamma_1\gamma_2}{\gamma_1 \cosh^2\left(\frac{\sqrt{\gamma_1\gamma_2}}{8}t\right) + \gamma_2 \sinh^2\left(\frac{\sqrt{\gamma_1\gamma_2}}{8}t\right)}$$

where a_2, γ_1, γ_2 are constants that can be calculated numerically (see [21]). Also, in both cases the argument of z will be given by

$$\arg z(t) = \delta_h^{(\pm)}(t) = \sin^{-1}\left(\frac{2}{B_{\mathrm{cr}}}\frac{d\sqrt{a_h^{(\pm)}(t)}}{\mathrm{d}t}\right). \tag{28}$$

A regular perturbation analysis with respect to the damping parameter (which is assumed to be weak, $\hat{\lambda} = \varepsilon^{1/2}\tilde{\lambda}$) reveals that these two homoclinic loops govern the global dynamics of the slow flow even in the presence of damping; moreover, they define the regime of optimal TET. Specifically, for a weakly damped system, optimal TET is realized for initial energies that correspond to trajectories that are in the neighborhood of the homoclinic loop. In particular the perturbed homoclinic orbit (due to weak damping) defines the boundary that separates trajectories that perform large excursions in the slow-flow phase space (leading to strong dissipation according to (26)) from those that are 'trapped' to the S11− region and lead to slow dissipation of energy (Fig. 3).

Moreover, from those trajectories that perform large excursions we can distinguish two different cases. In the first case the trajectory is initiated so close to the perturbed homoclinic orbit that the major part of the system energy is consumed rapidly while the system remains close to the homoclinic orbit, leading to optimum TET. In the second case the energy is sufficiently large so that even though the trajectory passes close to the perturbed homoclinic orbit, it reaches the S11+ regime (Fig. 3) and starts performing oscillations around it leading to dissipation of energy through nonlinear beats ('wiggles') (see [21]). Note that for all the above cases the slow-flow model is valid only for the initial stage of the dynamics and cannot predict the eventual transition to the low energy stage of the oscillation (where the system behaves linearly).

5.2. Targeted energy transfer in the presence of noise

We will now proceed to the analysis of the stochastic, damped transitions. Our study will be based on the numerical solution of the FPK equation using a finite-difference scheme. The technical details on the numerical scheme used for the solution of the FPK are given in [42]. Alternative methods for the analysis of this stochastic dynamical system may also be used. These include methods based on stochastic averaging and statistical linearization [43] or methods based on numerical path integration [44]. Both of these approaches rely explicitly on the Markovian assumption, are efficient computationally, and easily applicable even for cases of non-stationary processes with non-constant diffusion coefficients.

In the plots that follow we present the stochastic response in terms of the probability density function $\chi(z_1, z_2, t)$ superimposed with the deterministic solution (shown in white and black in the left and right plots, respectively). We also show in a separate plot the probability density function $\chi(|z|, t)$ for the modulus of the slow variable which is connected with the amount of energy that is dissipated (Eq. (26)). All the results presented correspond to $\varepsilon = 0.05$, $\hat{\lambda} = 0.1$, $\hat{C} = \frac{8}{3}$ and diffusion coefficient $\frac{\nu^2}{8} = 2 \times 10^{-2}$ except if stated otherwise. Note that the value of the diffusion coefficient ($\nu = 4 \times 10^{-1}$) has been chosen to be of the same order

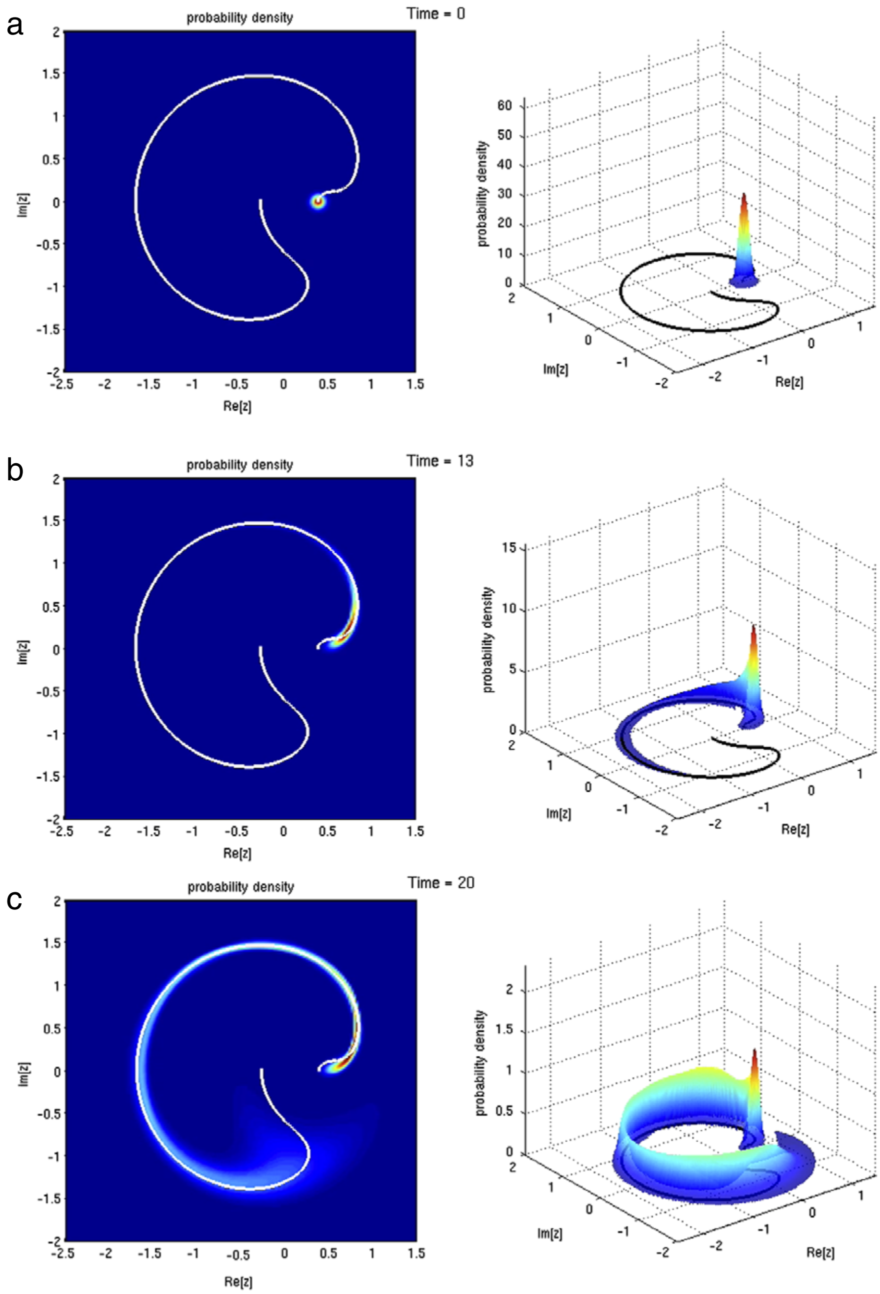

Fig. 4. Probability density function describing the slow-flow stochastic, damped transitions in the optimal TET regime (presented for different time instants).

as the response of the deterministic system near the optimal TET regime as this is predicted by the deterministic analysis.

5.2.1. Optimal TET regime

To study the stochastic transitions in the optimal TET regime we initiate the pdf on the critical value $B_{cr}(\hat{\lambda})$ predicted by the deterministic analysis (Fig. 4(a)). In Fig. 4(b) we observe the stretching of the probability measure along the geometry imposed by the outer homoclinic loop. Through this process, larger amounts of probability are moving to regions of large $|z|$ causing the pdf for $|z|$ to move towards higher values (see Fig. 5 for $t < 10 - 15$), consistent with the deterministic theory (black solid curve in Fig. 5).

As time evolves the stretching of the probability measure along the outer homoclinic orbit continues, with diffusion causing an almost uniform distribution of probability along the complete length of the outer homoclinic loop (Fig. 4(c)). As it is shown in Fig. 5 ($t \sim 15$) this process results in very large amounts of mean energy dissipation (because of the concentration of probability around large values of $|z|$). The governing mechanism in this case is the combined action of strong stretching (since the period of the homoclinic orbit is infinite and, therefore, the stretching occurring close to it is maximum) which enhances the process of diffusion causing a more rapid transfer of probability along the homoclinic loop and finally a uniform distribution along it. This mechanism results in a peak probability around large values of $|z|$ (see Fig. 5 for $t \sim 17$) which reveals its robustness since 'most' of the trajectories will reach this high value of $|z|$ at $t \sim 17$. This is the point of maximum dissipation since the mean-square distance from the origin $E^{\varpi}\left[|z(t;\varpi)|^2\right]$ is maximum.

The last strong dissipation regime is followed by a rapid decrease of the probability density function occurring uniformly

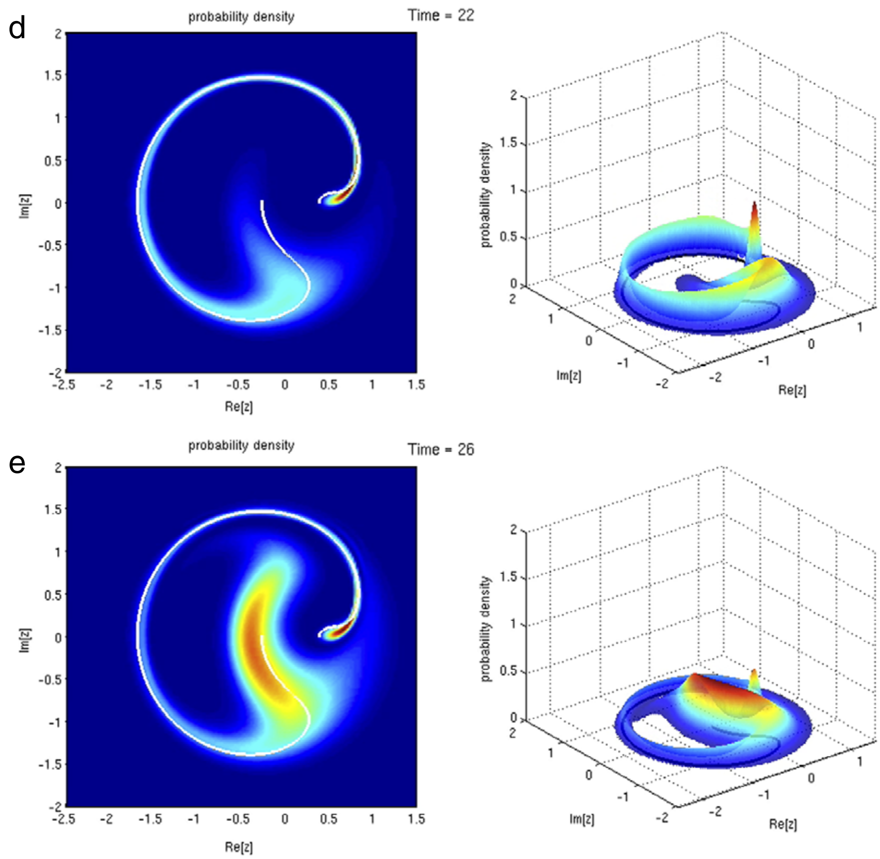

Fig. 4. (*continued*)

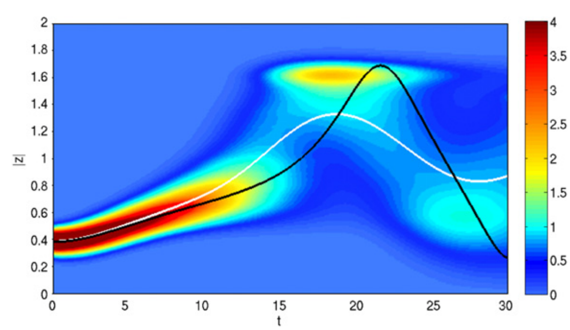

Fig. 5. Probability density function of the modulus $|z(t;\varpi)|$ with respect to time. The black solid line shows the deterministic response and the white solid line the mean $E^{\varpi}[|z(t;\varpi)|]$.

along the length of the outer homoclinic loop (Fig. 5(d)–(e)) and a simultaneous transfer of probability in regions of small $|z|$. This is also observed in Fig. 5 ($t \sim 20 - 25$) where we see the transport of the probability density function towards smaller values of $|z|$.

Note that during this stochastic transition some probability has been 'trapped' into the S11-region because, as we mentioned, the initial conditions are normally distributed along the critical value. Therefore, some amount of probability remains in the domain of attraction for the low energy dynamics (inefficient regime).

5.2.2. *Supercritical TET regime*

We now consider an initial probability density function initialized in the supercritical TET regime as predicted by the deterministic analysis. In this case, since we are further away from the homoclinic orbit, stretching is smaller, and the governing mechanism in this case is the translation of the probability measure along the direction of the homoclinic orbit (Fig. 6(b)). In this way the probability measure performs a full excursion around the outer homoclinic loop and ends up in the S11+ region (Fig. 6(c)) consistent with the deterministic orbit. At this point the oscillations ('wiggles' [21]) around the S11+ region begin (Fig. 6(d)–(e)) and last until the energy is dissipated and the system ends up in the low-energy regime (this transition is not captured by this early approximation model).

These observation are consistent with the probability density function $\chi(|z|, t)$ shown in Fig. 7. Note that in this case the deterministic and the mean stochastic dynamics are very close, a result that is expected if we take into account the common mechanism that governs the two cases; i.e., in this case diffusion doesn't interact with nonlinearity as in the optimal TET regime. Also, similar to the previous case, we observe that the extremes of the $E^{\varpi}[|z(t;\varpi)|]$ curve have concentrated probability which shows the robustness of the mechanism, since most of the trajectories will pass over these points.

Finally, we note that in contrast with the previous case no amount of probability is trapped into the suboptimal region since the initial probability is defined sufficiently far from the corresponding domain of attraction.

5.2.3. *Suboptimal TET regime*

Similar to the previous cases, we initiate the probability density function in the suboptimal regime (as defined by the deterministic analysis). The results show a 'trapping' of the probability density

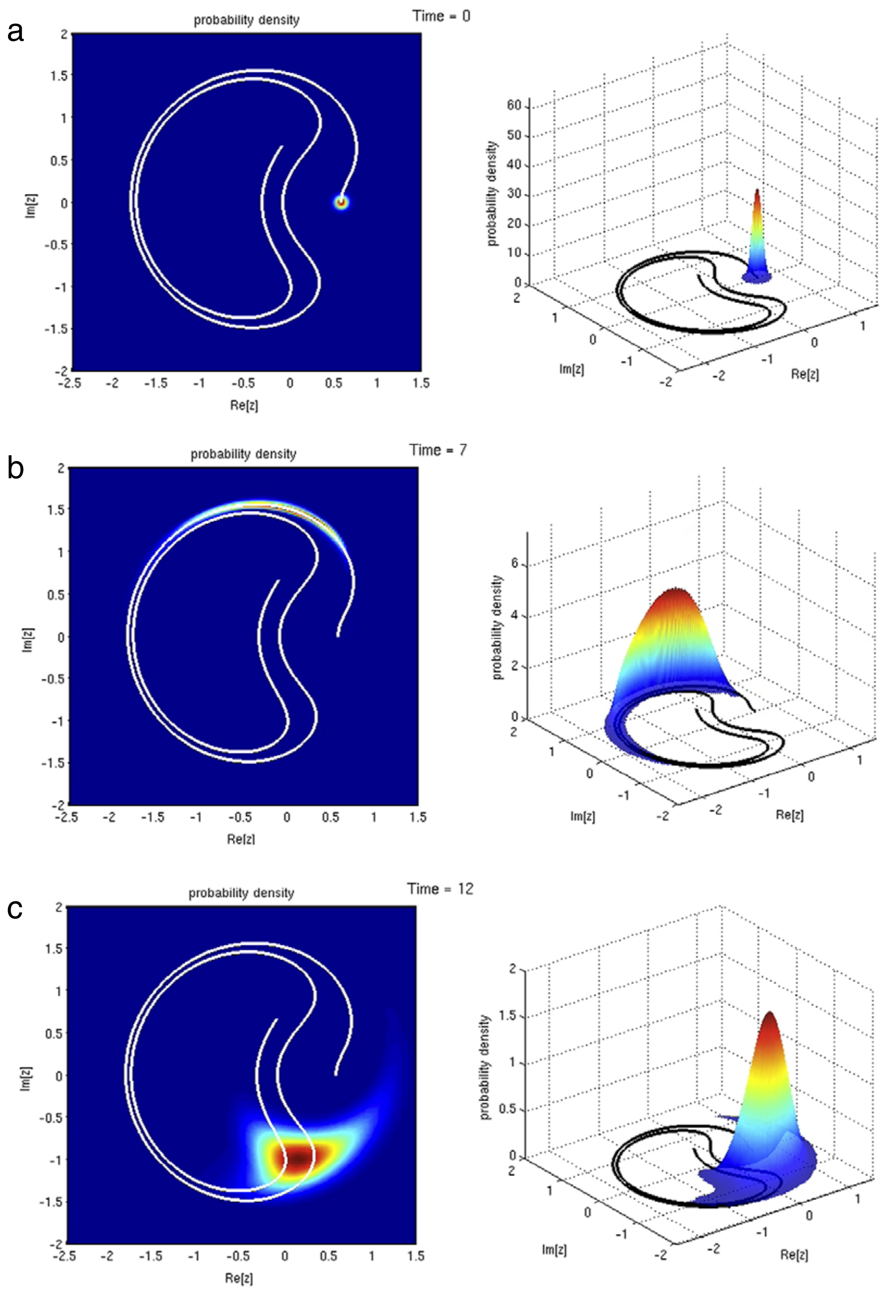

Fig. 6. Probability density function describing the slow-flow stochastic, damped transitions in the supercritical TET regime (presented for different time instants).

function in the low $|z|$ regime (Fig. 8) causing slow dissipation of the system energy. The same results are shown in Fig. 9 in terms of the probability density function for the modulus $|z|$. We emphasize that the constant value of $E^{\varpi}\left[|z\,(t;\varpi)|\right]$ corresponds to a mean decay of energy observed in a linear system.

5.3. Robustness of stochastic targeted energy transfer

In this section we study the robustness of the optimal and supercritical TET regimes by measuring the mean instantaneous energy that is dissipated

$$\bar{\epsilon}_d(t) = \frac{\varepsilon\hat{\lambda}}{2}E^{\varpi}\left[|z\,(t;\varpi)|^2\right]$$

as well as the percentage of the initial energy that remains in the system after time t

$$\bar{\alpha}(t) = 1 - \frac{\varepsilon\hat{\lambda}E^{\varpi}\left[\int_0^t |z\,(t;\varpi)|^2\,\mathrm{d}t\right]}{B^2} = 1 - \frac{\varepsilon\hat{\lambda}\int_0^t E^{\varpi}\left[|z\,(t;\varpi)|^2\right]\mathrm{d}t}{B^2}.$$

In Fig. 10 we present these two measures of robustness and efficiency for three different values of diffusion (noise intensity). In the first case (Fig. 10(a)) we have the deterministic system, where we observe from both measures the well defined separation of the optimal TET regime from the linear dissipation region. After the addition of a small amount of noise (Fig. 10(b)), we see that this transition becomes much smoother while the optimal TET region efficiency remains invariant. Therefore, in the presence of noise the optimal TET regime is not only maintained but it is enhanced and becomes more robust due to the interaction of

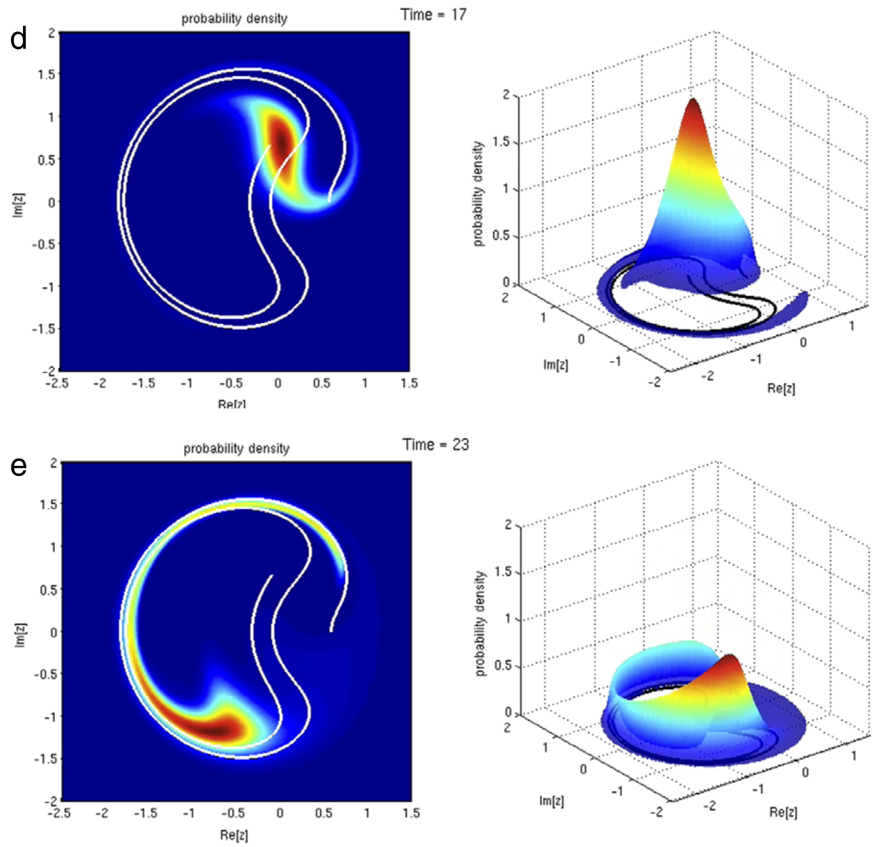

Fig. 6. (*continued*)

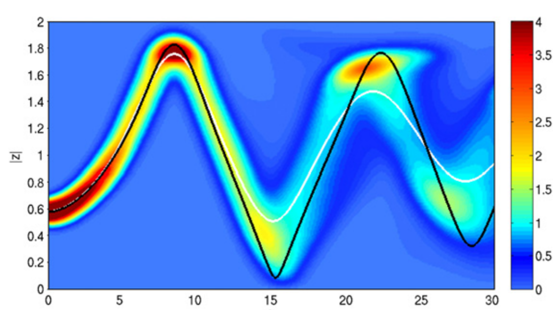

Fig. 7. Probability density function of the modulus $|z\,(t;\,\varpi)|$ with respect to time in the supercritical TET regime. The black solid line shows the deterministic response and the white solid line the mean $E^{\varpi}\,[|z\,(t;\,\varpi)|]$.

the stretching occurring near the homoclinic orbit (deterministic nonlinear dynamics) with diffusion (uncertainty) as described in the optimal TET section. As we increase the uncertainty of the system (Fig. 10(c)), we observe that the three distinct domains of TET performance begin to homogenize with an even smoother transition from the sub-optimal TET regime to the optimal TET. At the same time, the maximum energy dissipation (occurring close to $B = 0.4$) does not decrease.

5.3.1. *Stochastic initial conditions—probability of escape from the homoclinic orbit*

The analysis presented so far describes the stochastic transitions with deterministic initial conditions. Note that for the case of stochastic initial conditions, described through the random variable $B\,(\varpi)$, the FPK equation cannot be used directly since the stochastic differential equation (24) contains the random parameter $B_1 \equiv B\,(\varpi)$. To overcome the above technical difficulty and characterize the robustness of the TET mechanism, we note that the probability of the state $\chi\,(z_1, z_2, t \mid B\,(\varpi))$ given the stochastic characteristics of the random variable $B\,(\varpi)$ through the probability density function $\chi_B(B)$, will be given by

$$\chi\,(z_1, z_2, t \mid B\,(\varpi)) = \int \chi\,(z_1, z_2, t \mid B_0)\,\chi_B\,(B_0)\,\mathrm{d}B_0 \qquad (29)$$

where $\chi\,(z_1, z_2, t \mid B_0)$ is the probability density function characterizing the state of the system when the initial conditions are deterministic and equal to B_0. For the determination of $\chi\,(z_1, z_2, t \mid B_0)$, we use the FPK equation as before.

Using the expression (29) we can then characterize the robustness for any given distribution $\chi_B(B)$ by calculation of the energy dissipated. An alternative measure that can be used to characterize robustness of energy dissipation is the probability of escape $P_{esc}\,(B_0)$ from the inner homoclinic loop D_h defined by Eqs. (27), (28) (Fig. 2 (lower plot)). To calculate this probability we allow the system to evolve for a time interval of $\mathcal{O}\left(\hat{\lambda}^{-1}\right)$ and then compute the probability that it remains 'trapped' in the homoclinic loop D_h. Therefore, we have

$$P_{esc}\,(B_0) \equiv 1 - \int_{D_h} \chi\,(z_1, z_2, t \mid B_0)\,\mathrm{d}z_1\mathrm{d}z_2, \quad t \sim \mathcal{O}\left(\hat{\lambda}^{-1}\right). \quad (30)$$

Using $P_{esc}\,(B_0)$ we can calculate the probability of escape for any set of random initial conditions $B\,(\varpi)$ using Eq. (29)

$$P_{esc} = \int P_{esc}\,(B_0)\,\chi_B\,(B_0)\,\mathrm{d}B_0.$$

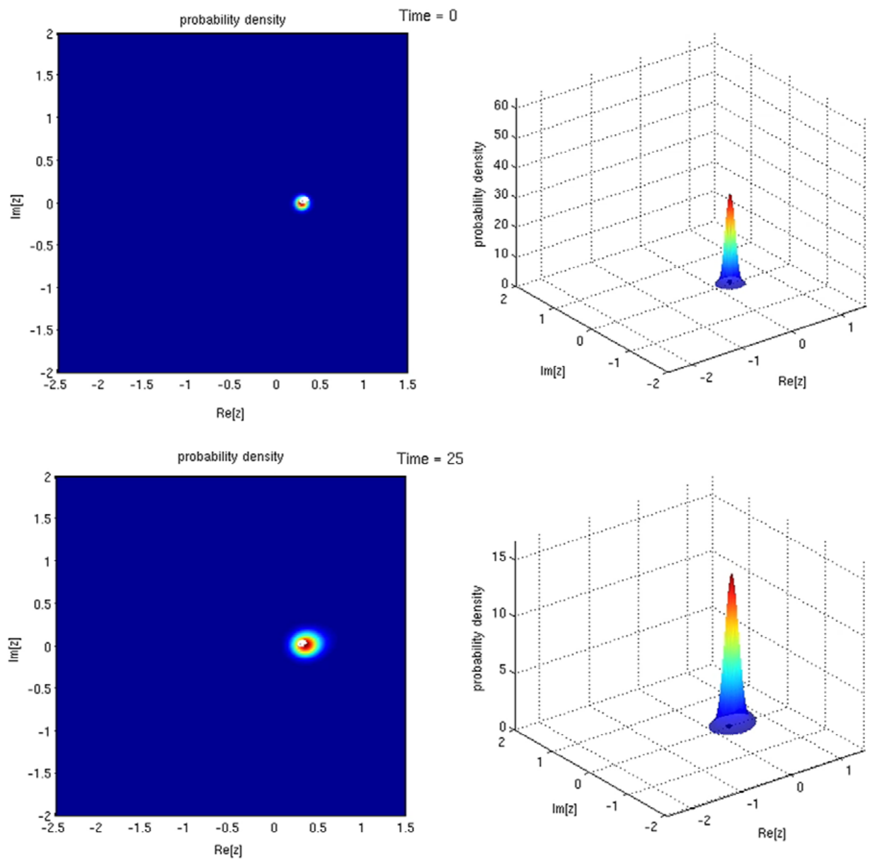

Fig. 8. Probability density function describing the slow-flow stochastic, damped transitions in the suboptimal TET regime (presented for different time instants).

In Fig. 11 we present the probability of trapping in the linear dissipation regime as computed through Eq. (30). We observe that the regions of suboptimal and optimal TET indeed can be characterized by this probability; however, this is not the case for the supercritical TET regime where the probability of escape remains uniform and equal to 1. Thus, it cannot be connected with the amount of dissipation in this region. This is expected if we take into account that the mechanism responsible for dissipation in the optimal and supercritical regime is the stretching and transport along the homoclinic orbit occurring away from the S11− region.

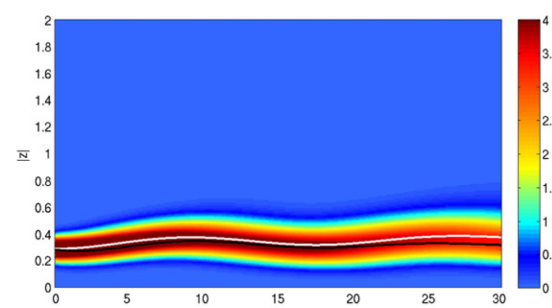

Fig. 9. Probability density function of the modulus $|z(t;\varpi)|$ with respect to time in the suboptimal TET regime. The black solid line shows the deterministic response and the white solid line the mean $E^{\varpi}[|z(t;\varpi)|]$.

Conclusions

In this work the problem of nonlinear interactions between a general linear structure and an essentially nonlinear attachment has been considered in a stochastic framework. Through a Green's function formulation we have shown that the full dynamics can be equivalently described by a single integro-differential equation. By restricting our analysis to the early time regime and subsequently applying complexification-averaging combined with the diffusion approximation, we derive a complex SDE that governs the system slow flow at its early stage. This equation is accompanied by analytical conditions which set the limits of its validity in terms of the linear subsystem and stochastic excitation characteristics.

We apply the derived SDE to the analysis of the stochastic slow dynamics of a single linear mode interacting with a nonlinear attachment. By considering the case of stochasticity comparable with the system response, we numerically solve the corresponding FPK equation and conclude that the optimal TET regime predicted by the deterministic analysis is not only preserved in the presence of stochasticity but is also enhanced due to the complex interaction of the diffusion in phase space (due to the presence of noise) and nonlinear advection. The basic nonlinear mechanisms such as the homoclinic orbit and the nonlinear beats observed in the deterministic analysis are also found in the stochastic context illustrating the robustness of the TET mechanism.

Future research directions include the application of the developed framework to the stochastic analysis of more complex systems such as interactions between closely spaced linear modes with a nonlinear attachment met in linear chains coupled with nonlinear oscillators. The analysis may also be useful for analyzing the problem of passive nonlinear energy harvesting from a stochastic environment such as water waves or general ambient vibrations in engineering systems.

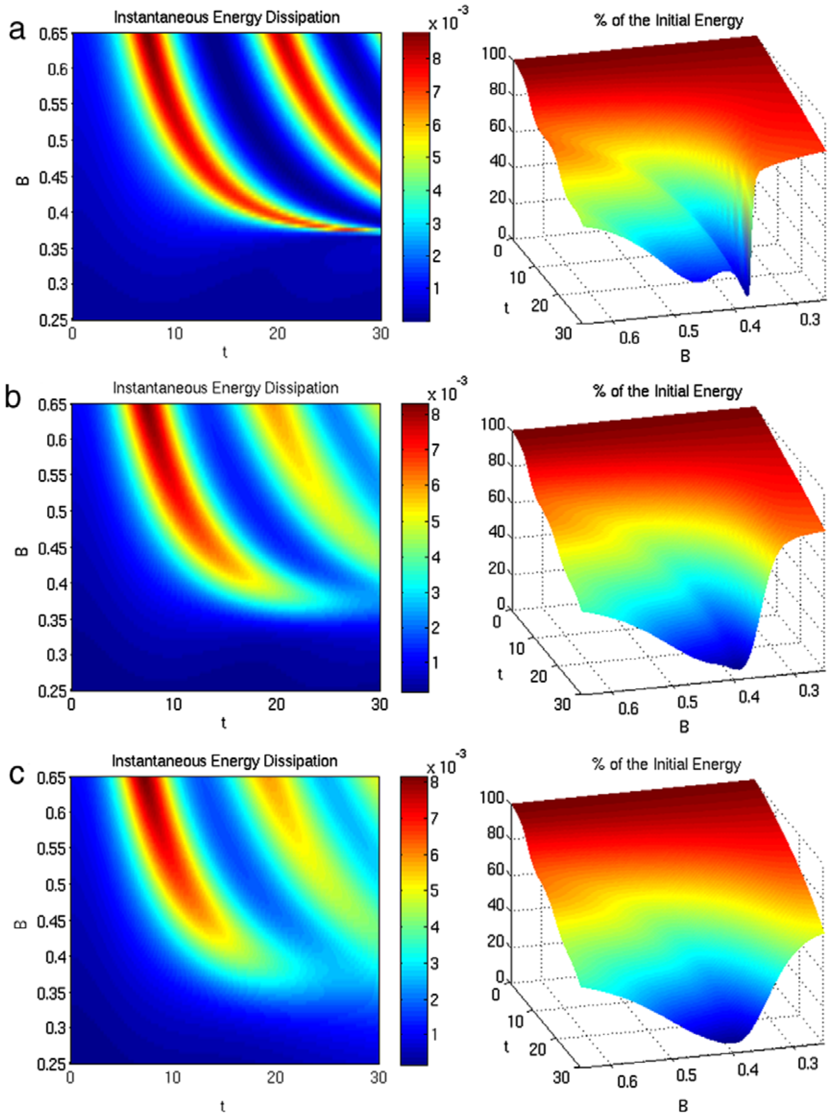

Fig. 10. Instantaneous energy dissipation from the nonlinear attachment and percentage of the initial energy that remains into the system after time t for (a) zero diffusion, (b) $\frac{\nu^2}{8} = 2 \times 10^{-2}$, and (c) $\frac{\nu^2}{8} = 4 \times 10^{-2}$.

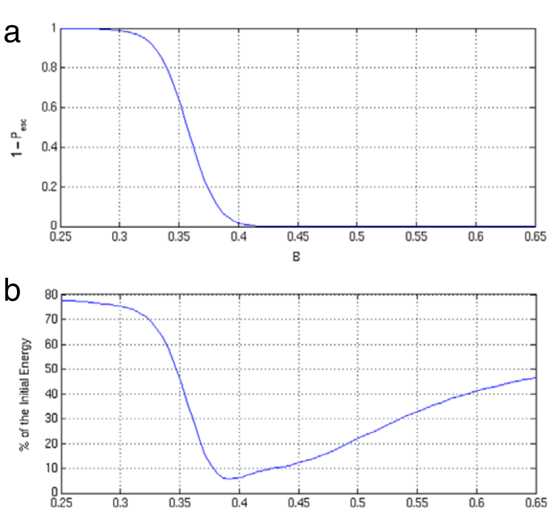

Fig. 11. (a) Probability that remains in the linear dissipation regime for $\frac{\nu^2}{8} = 2 \times 10^{-2}$ at $t = 30$. (b) Percentage of the initial energy that remains in the system at $t = 30$.

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