Kinetic and Related Models
©American Institute of Mathematical Sciences
Volume **13**, Number **4**, August **2020**

doi:10.3934/krm.2020024

pp. **703–737**

STRONG SOLUTIONS FOR THE ALBER EQUATION AND STABILITY OF UNIDIRECTIONAL WAVE SPECTRA

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(Communicated by Paola Pietra)

Abstract. The Alber equation is a moment equation for the nonlinear Schrödinger equation, formally used in ocean engineering to investigate the stability of stationary and homogeneous sea states in terms of their power spectra. In this work we present the first well-posedness theory for the Alber equation with the help of an appropriate equivalent reformulation. Moreover, we show linear Landau damping in the sense that, under a stability condition on the homogeneous background, any inhomogeneities disperse and decay in time. The proof exploits novel L^2 space-time estimates to control the inhomogeneity and our result applies to any regular initial data (without a mean-zero restriction). Finally, the sufficient condition for stability is resolved, and the physical implications for ocean waves are discussed. Using a standard reference dataset (the "North Atlantic Scatter Diagram") it is found that the vast majority of sea states are stable, but modulationally unstable sea states do appear, with likelihood $O(1/1000)$; these would be the prime breeding ground for rogue waves.

2010 *Mathematics Subject Classification.* Primary: 35Q35, 35B35; Secondary: 81S30.

Key words and phrases. Wigner transform, Alber equation, Landau damping, modulation instability, Penrose condition, ocean wave spectra.

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1. **Introduction.** The Alber equation

$$\begin{aligned}
&\partial_t w + 2\pi p k \partial_x w \\
&-qi \int_{\lambda, y \in \mathbb{R}^d} e^{-2\pi i \lambda y} \Big[n(x - \frac{y}{2}, t) - n(x + \frac{y}{2}, t) \Big] dy \, P(k - \lambda) d\lambda \\
&-\epsilon q i \int_{\lambda, y \in \mathbb{R}^d} e^{-2\pi i \lambda y} \Big[n(x - \frac{y}{2}, t) - n(x + \frac{y}{2}, t) \Big] dy \, w(x, k - \lambda, t) d\lambda = 0, \\
&\qquad n(x,t) = \int_{\xi \in \mathbb{R}^d} w(x, \xi, t) d\xi, \qquad w(x, k, 0) = w_0(x, k),
\end{aligned} \tag{1}$$

see e.g. [1, 4, 13, 23, 28, 29, 32, 36], is a second moment of the cubic NLS equation

$$i\partial_t u + \frac{p}{2} \Delta u + \frac{q}{2} |u|^2 u = 0 \tag{2}$$

governing the complex envelope of ocean waves, $u(x,t)$. It is derived by taking the stochastic second moment of equation 2 and then using a Gaussian moment closure. Then, passing to Wigner transform coordinates, the initial data is assumed to be close to a stationary and homogeneous background solution $P(k)$,

$$W(x,k,t) = P(k) + \epsilon w(x,k,t), \qquad \epsilon \ll 1.$$

Observe that the unknown $w(x,k,t)$ in equation 1 is the inhomogeneity $w(x,k,t)$.

The heuristic derivation of the Alber equation 1 from the NLS equation 2 is well-known, but for completeness it is outlined in Appendix B. The Gaussian closure means this is not an exact equation for the second stochastic moment. However, in the ocean waves context $q = o(1)$, $1/p = o(1)$, i.e. the problem is inherently weakly non-linear. This is a factor behind the empirical fact that the Gaussian closure is a meaningful one in this context [18, 22].

The power spectrum $P(k)$ represents the distribution of wave energy over wavenumbers in a homogeneous sea state. Typically one would expect that inhomogeneities disperse, thus preserving the leading-order stationary and homogeneous character of the wavefield; indeed, this is what we find for the vast majority of plausible sea states in Section 8. This would be the "Landau damping" / stable regime. However, in those exceptional cases where the inhomogeneity $w(x,k,t)$ is allowed to feed on the (infinite) energy of the power spectrum and grow significantly, then localized extreme events such as Rogue Waves become possible [4, 8, 10, 11, 13, 14, 23, 25, 29]. This would be the "modulation instability" (MI) / unstable regime, and it can be thought of as a generalization of the standard MI of the NLS [7, 37, 38] to continuous spectra.

The criterion for (in)stability involves only the power spectrum $P(k)$, and is related to the "eigenvalue relation" which appeared in [1] as a sufficient condition for instability. Some refinements are required, and the relation between the different kinds of (in)stability conditions is the object of Theorem 3.5 (see also Remark 3.1). A key fact here is that the bifurcation from Landau damping to MI involves only the shape of the power spectrum $P(k)$, and is not sensitive to the initial inhomogeneity, $w_0(x,k)$. Determining whether a power spectrum is stable or unstable is a crucial question in the oceanographic context, and presents certain challenges [13, 29, 32]. This is discussed in some detail in Section 8, where Theorem 3.5 is used in a novel, straightforward way to check the stability of a given spectrum $P(k)$.

While the Alber equation has been used formally for some time, there are still many open questions related to it. It has only been recently that some works for well-posedness and stability of related nonlinear equations have appeared. In [20, 19] the authors work in operator formalism, for a defocusing problem ($p \cdot q < 0$) with a regular interaction kernel[1]. The authors exploited the defocusing character of the problem by defining a relative entropy which controls the solution in an appropriate sense; this is a key ingredient of their proof. In [9] a similar argument is used for the defocusing problem with a δ interaction kernel and with a single background spectrum. Another related work is [34], where the stability of a fully stochastic problem (no Gaussian closure) is studied, but only in the defocusing case, $d \geqslant 4$ and with a smooth interaction kernel.

More broadly, there are analogies between the classical Landau damping problem for the Vlasov equation [21, 26] and the stability of the Alber equation. The most closely related work from that context seems to be [6] where the Vlasov equation is studied in $d = 3$ and with mean-zero initial data, as opposed to $d = 1$ (leading to weaker dispersion) and general initial data, which is the natural setting for the Alber equation.

This paper is organized as follows: in Section 2 some definitions and notations are summarized. The main results are formulated in Section 3. We show well-posedness and regularity of solutions for any dimension in Theorems 3.1, 3.2, 3.3 below. In Theorem 3.4 we consider the one-dimensional case and show for the first time that the inhomogeneities decay in time under a stability condition, using novel L^2 space-time estimates (cf. Lemmata 5.4 and 6.3). In Theorem 3.5 we derive the stability condition, and show it is complementary to the sufficient condition for instability, the"eigenvalue relation" mentioned earlier. Using Theorem 3.5 we investigate the stability of sea states in the North Atlantic in Section 8. The proofs of main results are given in Sections 4-7 and in Section 8 we discuss applications.

2. Mathematical preliminaries.

We shall start with summarizing main notations and definitions used in the statement and proofs of main results.

2.1. Definitions and notations.

The normalization we use for the Fourier transform is

$$\hat{u}(X) = \mathcal{F}_{x\to X}[u] = \int_{x\in\mathbb{R}^d} e^{-2\pi i x\cdot X} u(x)dx, \quad \check{u}(X) = \mathcal{F}^{-1}_{x\to X}[u] = \int_{x\in\mathbb{R}^d} e^{2\pi i x\cdot X} u(x)dx$$

for $X \in \mathbb{R}^d$.

Definition 2.1 (Spaces of bounded derivatives and moments)**.** Consider a function on phase-space $f(x,k)$, $s \in \mathbb{N}$, $p \in [1,\infty]$. The $\Sigma^{s,p}$ norm will be defined as

$$\|f\|_{\Sigma^{s,p}} = \sum_{0\leqslant|a+b+c+d|\leqslant s} \|x^a k^b \partial_x^c \partial_k^d f\|_{L^p(\mathbb{R}^{2d})}.$$

We will also use the standard Sobolev spaces

$$\|f\|_{W^{s,p}} = \sum_{0\leqslant|c+d|\leqslant s} \|\partial_x^c \partial_k^d f\|_{L^p}, \qquad \|f\|_{H^s} = \|f\|_{W^{s,2}}.$$

One readily checks the following

[1]That is, $K(x) * n(x,t)$ appears in the equation instead of $n(x,t)$. Whenever $n(x,t)$ appears directly, as in equation 1, this is also called a δ interaction kernel, since $n(x,t) = \delta(x) * n(x,t)$.

Lemma 2.2 (Embeddings of the $\Sigma^{s,p}$)**.** *By virtue of the Sobolev embeddings,*

$$\forall q,p\in[1,\infty],\ s\in\mathbb{N}\qquad \exists s_0\in\mathbb{N},\ C>0\qquad \|f\|_{\Sigma^{s,p}}\leqslant C\|f\|_{\Sigma^{s+s_0,q}}. \tag{3}$$

Denoting $\mathcal{S}(\mathbb{R}^{2d})$ the Schwarz class of test-functions on phase-space, observe that for any $q\in[1,\infty]$

$$\bigcap_{s\in\mathbb{N}}\Sigma^{s,q}(\mathbb{R}^{2d})=\mathcal{S}(\mathbb{R}^{2d}).$$

Moreover, the spaces $\Sigma^{s,2}$ are closed under Fourier transforms in the sense that

$$\mathcal{F}_{(x,k)\to(X,K)}\Sigma^{s,2}=\mathcal{F}_{k\to K}\Sigma^{s,2}=\mathcal{F}_{x\to X}\Sigma^{s,2}=\Sigma^{s,2}$$

and similarly for inverse Fourier transforms. Combined with equation 3, this means that

$$\forall q,p\in[1,\infty],\ s\in\mathbb{N}\qquad \exists s_0\in\mathbb{N},\ C>0\quad \textit{such that}\quad \|f\|_{\Sigma^{s,q}}\leqslant C\|\mathcal{F}f\|_{\Sigma^{s+s_0,p}}$$

where $\mathcal{F}$ denotes a forward or inverse Fourier transform in the x, k, or (x,k) variables.

We will also use the Laplace transform, denoted as

$$\widetilde{u}(\omega):=\mathcal{L}_{t\to\omega}[f]=\int_{t=0}^{+\infty}e^{-\omega t}u(t)dt,$$

and the Hilbert transform $\mathbb{H}$ and the signal transform $\mathbb{S}$

$$\mathbb{H}[u](x)=\frac{1}{\pi}\text{p.v.}\int_{t\in\mathbb{R}}\frac{u(t)}{x-t}dt,\qquad\qquad \mathbb{S}[u](x)=\mathbb{H}[u](x)-iu(x), \tag{4}$$

respectively.

In the context of the inverse Laplace transform we will also use an alternate "Fourier transform in time",

$$\mathfrak{F}_{t\to s}[u(t)]:=\int_{t\in\mathbb{R}}e^{-ist}u(t)dt,\qquad \mathfrak{F}^{-1}_{s\to t}[v(s)]=\frac{1}{2\pi}\int_{s\in\mathbb{R}}e^{ist}v(s)ds.$$

Obviously

$$\|\mathfrak{F}[u]\|_{L^2}=\sqrt{2\pi}\|u\|_{L^2},\qquad \|\mathfrak{F}^{-1}[v]\|_{L^2}=\frac{1}{\sqrt{2\pi}}\|v\|_{L^2},\qquad \mathfrak{F}[tu(t)]=i\partial_s\mathfrak{F}[u].$$

In the statement and proof of the main results we will also use the following

Definition 2.3 (D_XP)**.** For a function $P:\mathbb{R}\to\mathbb{R}$ we will use the notation

$$D_XP(k)=\begin{cases}\frac{P(k+\frac{X}{2})-P(k-\frac{X}{2})}{X}, & X\neq 0\\ P'(k), & X=0.\end{cases}$$

By abuse of notation all constants will be denoted by C,C',C''. To keep track of dependence on important parameters we will use e.g. $C=C(t,p,q)$.

2.2. **Reformulation of the problem and heuristics.** To study problem 1 it is helpful to use equivalent reformulations. If we take the inverse Fourier transform in x of the original Alber equation we pass to the Alber-Fourier equation

$$\begin{aligned} &\partial_t f - 4\pi^2 ipk\cdot Xf + qi\left[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\right]\check{n}(X,t) + \\ &\quad + \epsilon qi\int_{\mathbb{R}^d}\check{n}(y,t)\left[f\Big(X-y,k-\frac{y}{2}\Big) - f\Big(X-y,k+\frac{y}{2}\Big)\right]dy = 0, \\ &\check{n}(X,t) = \int_{\mathbb{R}^d} f(X,\xi,t)d\xi = \mathcal{F}^{-1}_{x\to X}[n(x,t)], \\ &f(X,k,0) = f_0(X,k) = \mathcal{F}^{-1}_{x\to X}[w_0], \end{aligned} \tag{5}$$

where

$$f(X,k,t) := \mathcal{F}^{-1}_{x\to X}[w(x,k,t)] = \int\limits_{\mathbb{R}^d} e^{2\pi i x\cdot X} w(x,k,t)dx.$$

To motivate linear stability, let us start from the linearized problem,

$$\begin{aligned} &\partial_t f - 4\pi^2 ipk\cdot Xf + qi\Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big]\check{n}(X,t) = 0, \\ &\check{n}(X,t) = \int_{\mathbb{R}^d} f(X,\xi,t)d\xi = \mathcal{F}^{-1}_{x\to X}[n(x,t)], \\ &f(X,k,0) = f_0(X,k) = \mathcal{F}^{-1}_{x\to X}[w_0]. \end{aligned} \tag{6}$$

By recasting in mild form we have

$$\begin{aligned} &f(X,k,t) - e^{4\pi^2 ipk\cdot Xt} f_0(X,k) = \\ &\quad = -qi\int\limits_0^t e^{4\pi^2 ipk\cdot X(t-\tau)}\Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big]\check{n}(X,\tau)d\tau, \end{aligned} \tag{7}$$

and by integrating in k we obtain a closed problem for $\check{n}(X,t)$,

$$\begin{aligned} &\check{n}(X,t) - \check{n}_f(X,t) = \int_{\tau=0}^t h(X,t-\tau)\check{n}(X,\tau)d\tau = 0, \\ &h(X,t) = 2q\,\sin(2\pi^2 pX^2 t)\check{P}(2\pi pXt), \end{aligned} \tag{8}$$

where $n_f(x,t)$ is the known "free-space position density",

$$\begin{aligned} &n_f(x,t) := \int\limits_{\mathbb{R}^d} w_0(x-2\pi pkt,k)dk \quad \Rightarrow \\ &\quad \Rightarrow \quad \check{n}_f(X,t) = \mathcal{F}^{-1}_{x\to X}[n_f(x,t)] = \int\limits_{\mathbb{R}^d} e^{4\pi^2 ipk\cdot Xt} f_0\,dk. \end{aligned} \tag{9}$$

Now denote for brevity

$$\widetilde{n}(X,\omega) := \mathcal{L}[\check{n}(X,t)], \quad \widetilde{n}_f(X,\omega) := \mathcal{L}[\check{n}_f(X,t)], \quad \widetilde{h}(X,\omega) := \mathcal{L}[h(X,t)]; \tag{10}$$

by taking the Laplace transform of equation 8 and rearranging terms we obtain

$$\begin{aligned} &\widetilde{n}(X,\omega) = \widetilde{n}_f(X,\omega) + \widetilde{h}(X,\omega)\widetilde{n}(X,\omega) \quad \Rightarrow \\ &\quad \Rightarrow \quad X\widetilde{n}(X,\omega) - X\widetilde{n}_f(X,\omega) = \frac{\widetilde{h}(X,\omega)}{1-\widetilde{h}(X,\omega)} X\widetilde{n}_f(X,\omega). \end{aligned} \tag{11}$$

This last equation will be the starting point for the proof of Theorem 3.4 in Section 6 (where the Laplace transforms will also be justified). For now it should clearly motivate the following

Definition 2.4 (Stability condition)**.** We will say that a spectrum $P \in \mathcal{S}(\mathbb{R})$ with compact support is stable if there is some $\kappa > 0$ such that

$$\inf_{\substack{\mathrm{Re}\,\omega>0,\\ X\in\mathbb{R}}} |1 - \widetilde{h}(X,\omega)| \geqslant \kappa > 0. \tag{12}$$

3. **Main results.** Here we state the main results of the paper.

Theorem 3.1 (Local well-posedness in L^1 for the Alber-Fourier equation)**.** *Let $f_0 \in L^1(\mathbb{R}^{2d})$, $P \in L^1(\mathbb{R}^d)$. Then there exists a maximal time*

$$T = T(\|f_0\|_{L^1(\mathbb{R}^{2d})}, q, \epsilon, \|P\|_{L^1(\mathbb{R}^d)}) > 0$$

such that there exists a unique mild solution $f(t) \in C([0,T), L^1(\mathbb{R}^{2d}))$ of equation 5.

Moreover, the blowup alternative holds, i.e.

$$\text{either } T = +\infty \text{ or } \lim_{t\to T^-} \|f(t)\|_{L^1(\mathbb{R}^{2d})} = +\infty.$$

The proof can be found in Section 4.1.

Theorem 3.2 (Higher regularity for solutions of the nonlinear problem)**.** *Denote $f(t)$ the solution of equation 5 with initial data $f_0 \in \mathcal{S}(\mathbb{R}^{2d})$, and T as in Theorem 3.1. Assume moreover that $P \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$f(t) \in \mathcal{S}(\mathbb{R}^{2d}) \qquad \forall t \in [0,T). \tag{13}$$

Moreover,

$$f \in C^\infty([0,T), \Sigma^{s,1}) \qquad \forall s \in \mathbb{N}. \tag{14}$$

Theorem 3.2 is proved in Section 4.2. Combined with Lemma 2.2 it yields local-in-time well-posedness and regularity of solutions for the Alber equation 1.

Theorem 3.3 (Global well-posedness and exponential bounds for the linearized problem)**.** *Denote $f(t)$ the solution of the linearized Alber-Fourier equation 6 with initial data $f_0 \in \mathcal{S}(\mathbb{R}^{2d})$. Assume moreover $P \in \mathcal{S}(\mathbb{R}^d)$. Then the maximal time is $T = +\infty$ for all initial data and for each $s \in \mathbb{N}$ there exists some $C = C(s,d,q,P) > 0$ so that*

$$\|f(t)\|_{\Sigma^{s,1}} \leqslant \|f_0\|_{\Sigma^{s,1}} C e^{Ct}. \tag{15}$$

Moreover, there exist some $s_2 = s_2(d)$ and $C = C(s_2,d,q,P)$ so that

$$\|\check{n}(t)\|_{L^\infty_X} + \|\partial_t \check{n}(t)\|_{L^\infty_X} + \|\partial_t f(t)\|_{L^\infty_{X,k}} \leqslant \|f_0\|_{\Sigma^{s_2,1}} C e^{Ct}. \tag{16}$$

The proof can be found in Section 4.2.

Theorem 3.4 (Landau damping for the Alber equation in $d = 1$)**.** *Let $P \in \mathcal{S}(\mathbb{R})$ be a background spectrum of compact support which is stable in the sense of Definition 2.4. Consider the linearized Alber equation*

$$\begin{aligned} &\partial_t w + 2\pi p k \cdot \partial_x w \\ &\quad - q\,i \int_{\lambda,y\in\mathbb{R}} e^{-2\pi i \lambda y} \Big[n\big(x + \frac{y}{2}, t\big) - n\big(x - \frac{y}{2}, t\big) \Big] dy\, P(k-\lambda)\, d\lambda = 0, \\ &n(x,t) = \int_{\xi\in\mathbb{R}} w(x,\xi,t) d\xi, \qquad w(x,k,0) = w_0(x,k) \in \mathcal{S}(\mathbb{R}^2). \end{aligned} \tag{17}$$

Then there exists $r \in \mathbb{N}$ large enough so that the force $\partial_x n(x,t)$ decays in time in the sense that

$$\|\partial_x n\|_{L^2_{x,t}} \leqslant C\frac{\kappa+1}{\kappa^2}\|w_0\|_{\Sigma^{r,\infty}}. \tag{18}$$

Furthermore, denoting $E(t): w_0(x,k) \mapsto w_0(x-2\pi pkt,k)$ the free-space propagator, there exists a wave operator $\mathbb{W}$ so that

$$\lim_{t\to\infty}\|w(t) - E(t)\mathbb{W}(w_0)\|_{L^\infty(\mathbb{R}^2)} = 0. \tag{19}$$

The proof is given in Section 6.

Theorem 3.5 (Equivalent formulations of the stability condition)**.** *Let $P(k) \in \mathcal{S}(\mathbb{R})$ be the background spectrum. Assume moreover that P is of compact support. Then the following statements are equivalent:*

(A). $\inf_{\substack{\mathrm{Re}\,\omega>0,\\ X\in\mathbb{R}}} |1-\widetilde{h}(X,\omega)| = 0$, *i.e. the spectrum is not stable in the sense of Definition 2.4.*

(B).

$$\exists\ X_* \in \mathbb{R},\quad \Omega_* \in \mathbb{C}\backslash\mathbb{R} \quad \textit{such that} \quad \mathbb{H}[D_{X_*}P](\Omega_*) = \mathbb{H}[D_{X_*}P](\overline{\Omega}_*) = \frac{4\pi p}{q}$$

or

$$\exists\ X_*, \Omega_* \in \mathbb{R} \quad \textit{such that} \quad \mathbb{H}[D_{X_*}P](\Omega_*) = \frac{4\pi p}{q} \quad \textit{and} \quad D_{X_*}P(\Omega_*) = 0.$$

(C). $d(\overline{\Gamma}, 4\pi p/q) = 0$, *where*

$$\begin{aligned} \Gamma_X &:= \{\mathbb{S}[D_X P(\cdot)](t),\ t\in\mathbb{R}\}\cup\{0\},\\ \overset{\circ}{\Gamma}_X &= \{z\in\mathbb{C} | z \textit{ enclosed by } \Gamma_X\},\quad \overline{\Gamma} := \overline{\bigcup_{X\in\mathbb{R}} \overset{\circ}{\Gamma}_X}. \end{aligned} \tag{20}$$

Moreover, we have the following sufficient condition for stability: if

$$\forall t_* \textit{ such that } D_X P(t_*) = 0 \textit{ the condition } \mathbb{H}[D_X P](t_*) < \frac{4\pi p}{q} \textit{ holds}$$

then P is stable in the sense of Definition 2.4.

The proof can be found in Section 7. An implementation of the criterion (C) above in the context of ocean engineering is visualized in Figures 3 and 4 and discussed in Section 8.

Remark 3.1 (Stability condition and Alber's nonlinear eigenvalue relation)**.** In [1] a two-dimensional setup is used, but the spectrum is integrated in the transverse direction, leading to an effective one-dimensional spectrum and a condition on that. This one-dimensional "eigenvalue relation" in our notation and scalings becomes

$$\exists X_* \in \mathbb{R},\ \mathrm{Re}(\omega_*) > 0 \quad \text{such that} \quad qi\int_{\mathbb{R}} \frac{P(k+\frac{X_*}{2}) - P(k-\frac{X_*}{2})}{\omega_* - 4\pi^2 ipk\, X_*} dk = 1. \tag{21}$$

If it is satisfied then linear instability follows. To see the relationship between this condition and (B) of Theorem 3.5 above observe that for $X_* \neq 0$, $\Omega_* := \omega_*/(4\pi p i X_*)$ equation 21 becomes

$$\exists 0 \neq X_* \in \mathbb{R},\ \Omega_* \in \mathbb{C} \ \text{ with } \ \mathrm{sign}(X_*)\cdot \mathrm{Im}(\Omega_*) < 0 \ \text{ such that}$$

$$\mathbb{H}[D_{X_*}P](\Omega_*) = \frac{4\pi p}{q}.$$

The form (B) in Theorem 3.5 appropriately takes into account the case $X_* = 0$ as well (equation 21 by construction has no solutions for $X_* = 0$, but stability may still fail due to what could be called renormalized solutions corresponding to $X_* = 0$).

Remark 3.2 (Compact support assumption for $P(k)$ in the main results)**.** In Theorem 3.4 the assumption that $P(k)$ has compact support is made. This allows for Theorem A.3 to be invoked in Section 6 in order to guarantee the integrability requirement of equation 66 in Theorem C.3, itself a central ingredient of the proof. The same assumption is also needed for Lemma A.2, which is invoked in the proof of Theorem 3.5. So it seems that in the current version of the proofs the compact support requirement cannot be removed, although this might eventually be possible with other techniques.

What does this mean in terms of the physical application in Section 8? Many widely used ocean power spectra involve power decay at infinity, $P(k) = O(|k|^{-a})$ for $|k| \to \infty$ [22], which technically is not of compact support. Even so, waves with wavenumber $|k| \gg 1$ would carry very little energy – and their physics would be predominantly surface tension and molecular effects, not hydrodynamics. So, from an ocean engineering point of view, applying a smooth cut-off to wavenumbers $|k| > K_M$ makes very little difference. Note furthermore that all the results would be uniform in K_M.

4. **Strong solutions for the Alber equation.** To simplify notations we can rewrite equation 5 as

$$\begin{aligned} \partial_t f - 4\pi^2 ipk \cdot Xf + \mathbb{B}[m, f] = 0, \\ m(X,t) = \int_{\mathbb{R}^d} f(X,k,t)dk, \qquad f(X,k,0) = f_0(X,k), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \mathbb{B}[m,f] = iq\Big[P\Big(k - \frac{X}{2}\Big) - P\Big(k + \frac{X}{2}\Big)\Big]m(X,t) + \\ + \epsilon iq \int_{\mathbb{R}^d} m(y,t)\Big[f\Big(X - y, k - \frac{y}{2}, t\Big) - f\Big(X - y, k + \frac{y}{2}, t\Big)\Big]dy. \end{aligned} \tag{23}$$

Lemma 4.1 (Bounds on $\mathbb{B}[m,f]$)**.** *Let $f, g, h \in L^1(\mathbb{R}^{2d})$, $m \in L^1(\mathbb{R}^d)$ and consider $\mathbb{B}[m,f]$ as defined in equation 23. Then*

$$\|\mathbb{B}[m,f]\|_{L^1(\mathbb{R}^{2d})} \leqslant 2|q|\,\|P\|_{L^1(\mathbb{R}^d)}\|m\|_{L^1(\mathbb{R}^d)} + 2|\epsilon q|\,\|m\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^{2d})} \tag{24}$$

and

$$\Big\|\mathbb{B}\Big[\int_{\mathbb{R}^d} f dk, f\Big]\Big\|_{L^1(\mathbb{R}^{2d})} \leqslant 2|q|\,\|P\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^{2d})} + 2|\epsilon q|\,\|f\|^2_{L^1(\mathbb{R}^{2d})}. \tag{25}$$

Moreover,

$$\begin{aligned} \Big\|\mathbb{B}\Big[\int_{R^d} g dk, g\Big] - \mathbb{B}\Big[\int_{\mathbb{R}^d} h dk, h\Big]\Big\|_{L^1(\mathbb{R}^{2d})} \leqslant \\ \leqslant 2|q|\left(\|P\|_{L^1(\mathbb{R}^d)} + |\epsilon|\,\|g\|_{L^1(\mathbb{R}^{2d})} + |\epsilon|\,\|h\|_{L^1(\mathbb{R}^{2d})}\right)\|g - h\|_{L^1(\mathbb{R}^{2d})}. \end{aligned} \tag{26}$$

Proof. For inequality 24 observe that

$$\|\mathbb{B}[m,f]\|_{L^1(\mathbb{R}^{2d})} \leqslant |q| \left\| \Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big] m(X,y) \right\|_{L^1(\mathbb{R}^{2d})} +$$
$$+ |\epsilon q| \left\| \int_{\mathbb{R}^d} m(y,t) \Big[f\Big(X-y, k-\frac{y}{2}, t\Big) - f\Big(X-y, k+\frac{y}{2}, t\Big)\Big] dy \right\|_{L^1(\mathbb{R}^{2d})}.$$

We will treat each term separately. Firstly,

$$\begin{aligned}
&\left\| \Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big] m(X,t) \right\|_{L^1(\mathbb{R}^{2d})} = \\
&\qquad = \int_{\mathbb{R}^{2d}} \left| P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big) \right| |m(X,y)| dX dk \\
&\qquad \leqslant \int_{\mathbb{R}^{2d}} \left| P\Big(k-\frac{X}{2}\Big) \right| |m(X,t)| dX dk + \int_{\mathbb{R}^{2d}} \left| P\Big(k+\frac{X}{2}\Big) \right| |m(X)| dX dk \\
&\qquad = \int_{\mathbb{R}^{2d}} \left| P\Big(k-\frac{X}{2}\Big) \right| dk |m(X,t)| dX + \int_{\mathbb{R}^{2d}} \left| P\Big(k+\frac{X}{2}\Big) \right| dk |m(X,t)| dX \\
&\qquad = 2\|P\|_{L^1(\mathbb{R}^d)} \|m\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^d} m(y,t) \Big[f\Big(X-y, k-\frac{y}{2}, t\Big) - f\Big(X-y, k+\frac{y}{2}, t\Big)\Big] dy \Big\|_{L^1(\mathbb{R}^{2d})} \\
&\qquad = \int_{\mathbb{R}^{3d}} |m(y,t)| \left| f\Big(X-y, k-\frac{y}{2}, t\Big) - f\Big(X-y, k+\frac{y}{2}, t\Big) \right| dy\, dX\, dk \\
&\qquad \leqslant \int_{\mathbb{R}^{3d}} \left| f\Big(X-y, k-\frac{y}{2}, t\Big) \right| dX\, dk\, |m(y,t)|\, dy + \\
&\qquad\qquad + \int_{\mathbb{R}^{3d}} \left| f\Big(X-y, k+\frac{y}{2}, t\Big) \right| dX\, dk\, |m(y,t)|\, dy \\
&\qquad = 2\|f(t)\|_{L^1(\mathbb{R}^{2d})} \|m(t)\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Combining the above inequality 24 follows. Inequality 25 follows by virtue of the elementary observation

$$\| \int_{\mathbb{R}^d} f dk \|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(X,k,t) dk \right| dX \leqslant \int_{\mathbb{R}^{2d}} |f(X,k,t)|\, dk dX = \|f\|_{L^1(\mathbb{R}^{2d})}.$$

For inequality 26 we expand

$$\begin{aligned}
&\mathbb{B}[\int_{\mathbb{R}^d} g dk, g] - \mathbb{B}[\int_{\mathbb{R}^d} h dk, h] = \\
&= iq\Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big]\Big(\int_{\mathbb{R}^d} g(X,k) dk - \int_k h(X,k) dk \Big) \\
&+ \epsilon iq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(y,k) dk \Big[g\Big(X-y, k-\frac{y}{2}\Big) - g\Big(X-y, k+\frac{y}{2}\Big)\Big] dy \\
&- \epsilon iq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(y,k) dk \Big[h\Big(X-y, k-\frac{y}{2}\Big) - h\Big(X-y, k+\frac{y}{2}\Big)\Big] dy \\
&+ \epsilon iq \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} g(y,k) dk - \int_{\mathbb{R}^d} h(y,k) dk \Big) \Big[h\Big(X-y, k-\frac{y}{2}\Big) - h\Big(X-y, k+\frac{y}{2}\Big)\Big] dy.
\end{aligned}$$

The result follows by treating each term as before. □

Moreover, consider some $s \in \mathbb{N}$ and the multi-indices $|\alpha+\beta+\gamma+\delta| \leqslant s$. Let us denote

$$f^{\alpha,\beta,\gamma,\delta} := X^\alpha k^\beta \partial_X^\gamma \partial_k^\delta f.$$

By direct computation one obtains

$$\begin{aligned} \partial_t f^{\alpha,\beta,\gamma,\delta} - 4\pi^2 ipk \cdot X f^{\alpha,\beta,\gamma,\delta} &= \mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f], \\ f^{\alpha,\beta,\gamma,\delta}(X,k,0) = f_0^{\alpha,\beta,\gamma,\delta}(X,k) &:= X^\alpha k^\beta \partial_X^\gamma \partial_k^\delta f_0(X,k). \end{aligned} \tag{27}$$

The detailed expression for $\mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f]$ can be found in Appendix D, and it contains terms of the form $f^{\alpha',\beta',\gamma',\delta'}$, $X^{\alpha'}\partial_X^{\gamma'} m(X,t)$, for $\alpha'+\beta'+\gamma'+\delta' \leqslant \alpha+\beta+\gamma+\delta$. Furthermore, one can directly – if somewhat tediously – obtain the following

Lemma 4.2 (Bound on the nonlinearity $\mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f]$)**.** *Let $1 \leqslant s \in \mathbb{N}$, and consider multi-indices*

$$|\alpha+\beta+\gamma+\delta| \leqslant s,$$

and $\mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f]$ as in Appendix D. Assume also that $P \in \mathcal{S}(\mathbb{R}^d)$. Then there exists a $C = C(s,d,q,P) > 0$ such that

$$\|\mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f]\|_{L^1(\mathbb{R}^{2d})} \leqslant C\Big[1 + \epsilon \|f\|_{\Sigma^{s-1,1}}\Big] \|f\|_{\Sigma^{s,1}}.$$

4.1. **Proof of Theorem 3.1.** Denote

$$U(t) : g(X,k) \mapsto e^{4\pi^2 ipk\cdot Xt} g(X,k), \tag{28}$$

the free-space propagator for the Alber-Fourier equation, i.e. $g(t) = U(t)g_0$ means that $\partial_t g - 4\pi^2 ipk \cdot Xg = 0$ and $g(0) = g_0$. Observe that, by construction, $\|U(t)g\|_{L^1_{X,k}} = \|g\|_{L^1_{X,k}}$. Equation 22 can now be written in mild form as

$$f(X,k,t) = U(t)f_0 - \int_0^t U(t-\tau)\mathbb{B}\Big[\int_{\mathbb{R}^d} f(\tau)dk, f(\tau)\Big]d\tau. \tag{29}$$

Define

$$\mathbb{E} := \big\{ g \in L^\infty(0,T_0; L^1(\mathbb{R}^{2d})) \text{ so that } \|g\|_{L^\infty(0,T_0;L^1(\mathbb{R}^{2d}))} \leqslant M \big\}$$

for some $M, T_0 > 0$, to be determined below. Moreover denote

$$\mathbb{G} : \mathbb{E} \ni g \mapsto U(t)f_0 - \int_0^t U(t-\tau)\mathbb{B}\Big[\int_{\mathbb{R}^d} g(\tau)dk, g(\tau)\Big]d\tau.$$

We will show that the operator $\mathbb{G}$ is a strict contraction on $\mathbb{E}$. First we need to show that $\mathbb{G}\mathbb{E} \subseteq \mathbb{E}$. Direct application of inequality 25 from Lemma 4.1 yields

$$\begin{aligned} \|\mathbb{G}g\|_{L^\infty(0,T_0;L^1(\mathbb{R}^d))} &\leqslant \|f_0\|_{L^1} + T_0 \Big\| B\Big[\int_{\mathbb{R}^d} g(\tau)dk, g(\tau)\Big]\Big\|_{L^\infty L^1} \\ &\leqslant \|f_0\|_{L^1} + T_0|q| \left(2\|P\|_{L^1}\|g\|_{L^\infty L^1} + 2|\epsilon|\|g\|^2_{L^\infty L^1}\right) \\ &\leqslant \|f_0\|_{L^1} + T_0|q|(2\|P\|_{L^1}M + 2|\epsilon|M^2). \end{aligned}$$

A (non-sharp) way to guarantee that $\|\mathbb{G}g\|_{L^\infty_t L^1_{X,k}} \leqslant M$ is to consider

$$M = 2\|f_0\|_{L^1(\mathbb{R}^{2d})} \quad \text{and} \quad T_0 < \frac{1}{|q|\ \max\{4\|P\|_{L^1(\mathbb{R}^d)},\ 4|\epsilon|\}\ (M+1)}. \tag{30}$$

Now using inequality 26 from Lemma 4.1, for any $g, h \in \mathbb{E}$ we obtain

$$\begin{aligned}\|\mathbb{G}g - \mathbb{G}h\|_{L^\infty L^1} &\leqslant T_0 \|B[\int_k g(\tau)dk, g(\tau)] - B[\int_k h(\tau)dk, h(\tau)]\|_{L^\infty_\tau L^1} \\ &\leqslant 2T_0|q| \left(\|P\|_{L^1(\mathbb{R}^d)} + |\epsilon| \, \|g\|_{L^1(\mathbb{R}^{2d})} + |\epsilon| \, \|h\|_{L^1(\mathbb{R}^{2d})} \right) \|g - h\|_{L^1(\mathbb{R}^{2d})} \\ &\leqslant 2T_0|q|(\|P\|_{L^1(\mathbb{R}^d)} + 2\epsilon M)\|g - h\|_{L^1(\mathbb{R}^{2d})}.\end{aligned}$$

For T_0 satisfying condition 30 the Lipschitz constant $L \leqslant T_0|q|(\|P\|_{L^1} + 2\epsilon M)$ of the mapping is strictly smaller than 1. Therefore, by virtue of the Banach Fixed Point Theorem, there exists a unique fixed point $f \in \mathbb{E}$, $f = \mathbb{G}f$, i.e. a unique mild solution of 22 for $t \in (0, T]$. Observe that by construction $\mathbb{G}g$ is continuous in time as a mapping with values in $L^1(\mathbb{R}^{2d})$.

Since $\|f(T_0)\|_{L^1(\mathbb{R}^{2d})} < \infty$, we can repeat the argument and extend the solution in time. Thus the blowup alternative follows, i.e. either the solution exists for all times, or there exists a finite blow-up time $T < \infty$ so that $\lim_{t \to T^-} \|f(t)\|_{L^1(\mathbb{R}^{2d})} = +\infty$. Whether T is finite or infinite, it will be called the *maximal time* for which $f(X, k, t)$ exists.

To show continuous dependence of solutions of 22 on initial data we consider $f(X, k, t)$ as above and $g(X, k, t)$ being a solution of 22 with initial data $g_0(X, k)$. Take some T_1 smaller than both the maximal times of f and g; then there exists some M_1 so that

$$\|f(t)\|_{L^\infty(0,T_1;L^1(\mathbb{R}^{2d}))}, \ \|g(t)\|_{L^\infty(0,T_1;L^1(\mathbb{R}^{2d}))} \leqslant M_1.$$

Now denote $h := f - g$; by subtracting the equations for f, g and using the same ideas as above, it follows that for all $t \in [0, T_1]$

$$\begin{aligned}\|h(t)\|_{L^1(\mathbb{R}^{2d})} \leqslant &\|f_0 - g_0\|_{L^1(\mathbb{R}^{2d})} + 2\int_0^t \|h(\tau)\|_{L^1} \left(\|P\|_{L^1(\mathbb{R}^d)} + \epsilon \|f(\tau)\|_{L^1(\mathbb{R}^{2d})} \right. \\ &\left. + \epsilon \|g(\tau)\|_{L^1(\mathbb{R}^{2d})} \right) d\tau \qquad (31) \\ \leqslant &\|f_0 - g_0\|_{L^1(\mathbb{R}^{2d})} + 2(\|P\|_{L^1(\mathbb{R}^d)} + 2\epsilon M_1) \int_0^t \|h(\tau)\|_{L^1(\mathbb{R}^{2d})} d\tau.\end{aligned}$$

Applying the the Gronwall inequality yields

$$\|h(t)\|_{L^1} \leqslant \|f_0 - g_0\|_{L^1} \left(1 + t2(\|P\|_{L^1} + 2\epsilon M_1) e^{t2(\|P\|_{L^1} + 2\epsilon M_1)} \right) \qquad \forall t \in [0, T_1],$$

and hence the continuous dependence of solutions on initial data. □

4.2. **Propagation of regularity and Proof of Theorems 3.2 & 3.3.**

Theorem 4.3 (Local well-posedness for the nonlinear Alber-Fourier-I equation on $\Sigma^{s,1}$)**.** *Denote $f(X, k, t)$ the solution of 22 with initial data $f_0(X, k) \in \Sigma^{s,1}$, $T = T(\|f_0\|_{L^1}, q, \epsilon, \|P\|_{L^1})$ the maximal time for which $f(t) \in L^1(\mathbb{R}^{2d})$ and $M^0(t) := \|f(t)\|_{L^1(\mathbb{R}^{2d})} \in C[0, T)$. Moreover, for each $1 \leqslant s \leqslant a_0$ denote $M^s(t) := \|f(t)\|_{\Sigma^{s,1}}$. Then there exists a constant $C > 0$ depending on s, d, q, ϵ, P and the background spectrum P such that*

$$M^s(t) \leqslant M^s(0) + C(s) \int_0^t M^{s-1}(\tau) M^s(\tau) d\tau \qquad \forall t \in [0, T), \qquad (32)$$

and therefore, for all $s \in \mathbb{N}$,

$$M^s(t) < \infty \quad \forall t \in [0, T), \qquad f(t) \in C([0, T), \Sigma^{s,1}). \qquad (33)$$

Proof. Consider multi-indices $|\alpha+\beta+\gamma+\delta| \leqslant s$; as was seen earlier, $f^{\alpha,\beta,\gamma,\delta} := X^\alpha k^\beta \partial_X^\gamma \partial_k^\delta f$ satisfies equation 27. By passing to mild form we have

$$f^{\alpha,\beta,\gamma,\delta}(t) = U(t) f_0^{\alpha,\beta,\gamma,\delta} + \int_0^t U(t-\tau)\mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f(\tau)]d\tau.$$

Taking L^1 norms and using Lemmata 4.1 and 4.2 we have

$$\|f^{\alpha,\beta,\gamma,\delta}(t)\|_{L^1} \leqslant \|f_0^{\alpha,\beta,\gamma,\delta}\|_{L^1} + C\int_0^t \|f(\tau)\|_{\Sigma^{s-1,1}}\|f(\tau)\|_{\Sigma^{s,1}}d\tau. \tag{34}$$

Equation 32 follows by summing over all $|\alpha+\beta+\gamma+\delta| \leqslant s$. The first part of equation 33 follows by applying recursively Gronwall's inequality to equation 32. The second part of equation 33 follows automatically from the mild form 34 since the time integrals now are known to exist. □

Proof of Theorem 3.2: For the proof of equation 13 it suffices to observe that the $\bigcap_{s\in\mathbb{N}} \Sigma^{s,1}(\mathbb{R}^{2d})$ regularity is propagated in time by virtue of Theorem 4.3, and that it implies Schwarz-class regularity by virtue of Lemma 2.2.

For the proof of smoothness with respect to the time variable stated in equation 14, observe that upon applying the operator ∂_t^l to equation 22, one obtains the problem

$$\partial_t(\partial_t^l f) - 4\pi^2 ipk\cdot X(\partial_t^l f) + \mathbb{B}[m,\partial_t^l f] = \mathbb{B}_{(l)}[f], \qquad m(X,t) = \int_{\mathbb{R}^d} f(X,k,t)dk,$$
$$\partial_t^l f(0) = 4\pi^2 ipk\cdot X(\partial_t^{l-1} f) - \mathbb{B}[m,\partial_t^{l-1} f] + \mathbb{B}_{(l-1)}[f],$$

where $\mathbb{B}_{(0)}[f] = 0$ and

$$\frac{i}{\epsilon q}\mathbb{B}_{(l)}[f] =$$
$$= \sum_{0\leqslant l'<l}\binom{l}{l'}\int_{\mathbb{R}^d}\partial_t^{l-l'} m(y,t)\Big[\partial_t^{l'} f\Big(X-y,k-\frac{y}{2},t\Big) - \partial_t^{l'} f\Big(X-y,k+\frac{y}{2},t\Big)\Big]dy.$$

By working recursively in l as in the proof of Theorem 4.3, the result follows. □

Proof of Theorem 3.3: We start by recasting equation 27 in mild form and taking the L^1 norm. Using the fact that $\epsilon = 0$ and Lemma 4.2 we obtain

$$\|f^{\alpha,\beta,\gamma,\delta}(t)\|_{L^1(\mathbb{R}^{2d})} \leqslant \|f_0^{\alpha,\beta,\gamma,\delta}\|_{L^1(\mathbb{R}^{2d})} + C\int_0^t \|f(\tau)\|_{\Sigma^{s,1}(\mathbb{R}^{2d})}d\tau.$$

Summing over all $|\alpha+\beta+\gamma+\delta| \leqslant s$ yields

$$\|f(t)\|_{\Sigma^{s,1}(\mathbb{R}^{2d})} \leqslant \|f_0\|_{\Sigma^{s,1}(\mathbb{R}^{2d})} + C\int_0^t \|f(\tau)\|_{\Sigma^{s,1}(\mathbb{R}^{2d})}d\tau.$$

Then estimate 15 follows by Gronwall's inequality.

By virtue of Lemma 2.2, for any r and for s' large enough we have $\|f(t)\|_{\Sigma^{r,\infty}} \leqslant C\|f(t)\|_{\Sigma^{s',1}} \leqslant Ce^{Ct}\|f_0\|_{\Sigma^{s',1}}$.

Now for the position density observe that

$$\begin{aligned}\|\breve{n}(t)\|_{L^\infty(\mathbb{R}^d)} &= \sup_{X\in\mathbb{R}^d} \Big|\int_{\mathbb{R}^d} f(X,\xi,t)d\xi\Big| \\ &\leqslant \int_{\mathbb{R}^d} \frac{d\xi}{\sup\limits_{X\in\mathbb{R}}(1+|\xi|^{d+1})} \sup_{X,\xi\in\mathbb{R}^d} |(1+|\xi|^{d+1})f(X,\xi,t)| \\ &\leqslant C\|f(t)\|_{\Sigma^{d+1,\infty}} \leqslant C' e^{C't}\|f_0\|_{\Sigma^{s',1}}.\end{aligned}$$

Moreover, considering equation 6 and using assumptions on P we obtain

$$\|\partial_t f(t)\|_{L^\infty(\mathbb{R}^{2d})} \leqslant \|k\cdot Xf(t)\|_{L^\infty(\mathbb{R}^{2d})} + C\|\breve{n}(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|f(t)\|_{\Sigma^{s_1,1}(\mathbb{R}^{2d})},$$

for some $s_1\in\mathbb{N}$ large enough. Similarly

$$\begin{aligned}\partial_t\breve{n} &= \int_{R^d} \partial_t f(X,k,t)dk \\ &= 4\pi^2 ip\int_{\mathbb{R}^d} k\cdot Xf dk - iq\int_{\mathbb{R}^d}\Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big]dk\int_{\mathbb{R}^d} f(X,\xi,t)d\xi\end{aligned}$$

implies

$$\|\partial_t\breve{n}(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant C\||k|^{d+2}|X|f(t)\|_{L^\infty(\mathbb{R}^{2d})} + C\|\breve{n}(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|f(t)\|_{\Sigma^{s_1',1}(\mathbb{R}^{2d})},$$

for some $s_1'\in\mathbb{N}$ large enough. Thus estimate 16 follows by selecting $s_2 = \max\{s', s_1, s_1'\}$. □

5. **The free-space position density.** In this Section we will establish some properties of the free-space position density $n_f(x,t)$, defined in equation 9, that we will use for the proof of Theorem 3.4.

Lemma 5.1 (Alternative expression for $\breve{n}_f.$)**.**

$$\breve{n}_f(X,t) := \mathcal{F}^{-1}_{x\to X}[n_f(x,t)] = \breve{w}_0(X, 2\pi ptX), \tag{35}$$

where $\breve{w}_0(A,B) = \mathcal{F}^{-1}_{(x,k)\to(A,B)}[w_0(x,k)]$.

Proof. Simple calculations yield

$$\begin{aligned}\breve{n}_f(X,t) &= \int_{\mathbb{R}} e^{2\pi i x\,X} n_f(x,t)dx = \int_{\mathbb{R}^2} e^{2\pi i x\,X} w_0(x-2\pi pkt,k)dkdx \\ &= \int_{\mathbb{R}^4} e^{2\pi i x\,X - 2\pi i[A\,(x-2\pi pkt)+B\,k]}\breve{w}_0(A,B)dkdxdAdB \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2\pi i x\,[X-A]} e^{2\pi i k\,[2\pi ptA-B]}dxdk\ \breve{w}_0(A,B)dAdB = \breve{w}_0(X,2\pi ptX).\end{aligned}$$

□

Lemma 5.2 (Uniform bound for $X\tilde{n}_f.$)**.** *Assume that there exists some $D>0$ such that*

$$|\breve{w}_0(X,K)| \leqslant \frac{D}{1+|X|^2+|K|^2}$$

and $\tilde{n}_f(X,\omega)$ as in equation 10. Then, there exists a constant $C>0$ such that for all $X\in\mathbb{R}$

$$\sup_{\mathrm{Re}\,\omega>0} |X\tilde{n}_f(X,\omega)| \leqslant C\,D.$$

Proof. Using Lemma 5.1 one readily checks that

$$\begin{aligned}\sup_{\mathrm{Re}\,\omega>0}|X\widetilde{n}_f(X,\omega)| &= \sup_{\mathrm{Re}\,\omega>0}\int_0^\infty |e^{-\omega t}X\breve{n}_f(X,t)|dt \leqslant \int_0^\infty |X\breve{n}_f(X,t)|dt\\ &= \int_0^\infty |X\breve{w}_0(X,2\pi ptX)|dt\\ &\leqslant \int_0^\infty \Big|X\frac{D}{1+|X|^2+|2\pi ptX|^2}\Big|dt \leqslant \int_0^\infty \frac{CD}{1+|2\pi pt|^2}dt.\end{aligned}$$

□

Observation 5.3. *We will use assumptions of the form*

$$|\breve{w}_0(X,K)| \leqslant \frac{D_r}{1+|X|^r+|K|^r}$$

in the sequel, which are weaker versions of $\breve{w}_0 \in \Sigma^{r,\infty}(\mathbb{R}^{2d})$. *By virtue of Lemma 2.2 it follows that, for some* r' *large enough*

$$D_r \leqslant \|\breve{w}_0\|_{\Sigma^{r,\infty}(\mathbb{R}^{2d})} \leqslant C\|w_0\|_{\Sigma^{r',\infty}(\mathbb{R}^{2d})}.$$

Lemma 5.4 (Space-time L^2 estimates for the free-space position density)**.** *Let*

$$|\breve{w}_0(X,K)| \leqslant \frac{D_r}{1+|X|^r+|K|^r}$$

for some large enough r *and constant* $D_r>0$. *Assume moreover* $r-\frac{1}{2}>a>b\geqslant 0$ *(*a,b,r *don't have to be integer.) Then*

$$\left(\int_{X,t}|X^a t^b\breve{n}_f(X,t)|^2dXdt\right)^{\frac{1}{2}} \leqslant C(a,b)D_r.$$

Proof. We will break up the norm as follows:

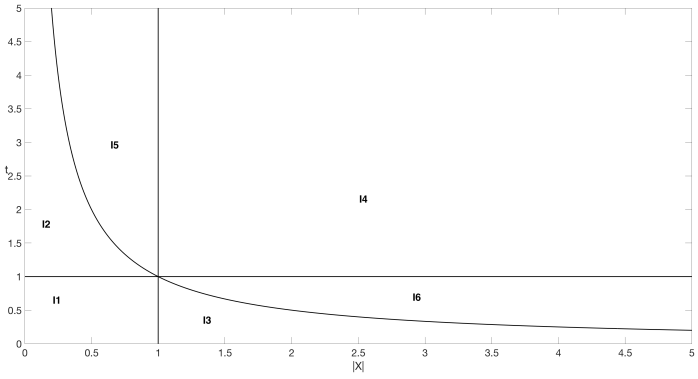

FIGURE 1. The domains of integration for the integrals I_j, $j=1,\dots,6$.

$$\begin{aligned}\|X^a t^b \breve{n}_f(X,t)\|_{L^2_{X,t}} &= \int\limits_{t,X} |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dX dt \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \\ &= \int\limits_{|X|<1, 0<t<1} + \int\limits_{|X|<1, 1<t<1/|X|} + \int\limits_{|X|>1, 0<t<1/|X|} \\ &\quad + \int\limits_{|X|>1, 1<t} + \int\limits_{1/t<|X|<1, 1<t} + \int\limits_{|X|>1, 1/|X|<t<1},\end{aligned}$$

see Figure 1. One readily observes that

$$\begin{aligned}I_1 &= \int_{|X|<1}\int_{|t|<1} |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dt dX \leqslant C D_r^2, \\ I_2 &= \int_{|X|<1}\int_1^{1/|X|} |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dt dX \leqslant D_r^2 C \int_{|X|<1} |X|^{2a} \int_1^{1/|X|} t^{2b} dt dX \\ &\leqslant D_r^2 C \int_{|X|<1} |X|^{2a-2b-1} dX \leqslant D_r^2 C, \\ I_3 &= \int_{|X|>1}\int_0^{1/|X|} |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dt dX \\ &\leqslant D_r^2 \int_{|X|>1}\int_0^1 \frac{|X|^{2a}}{|X|^{2r}} dt dX = C D_r^2 \int_1^\infty X^{2(a-r)} dX \leqslant C D_r^2.\end{aligned}$$

By using the elementary observation that $\frac{x^{2a}t^{2b}}{(1+x^r+(xt)^r)^2} \leqslant \frac{x^{2a}t^{2b}}{(xt)^{2r}} = x^{2(a-r)}t^{2(b-r)}$ for $t \geqslant 0$, $x \geqslant 0$, we have

$$\begin{aligned}I_4 &= \int_1^\infty \int_{|X|\geqslant 1} |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dt dX \\ &\leqslant C D_r^2 \int_1^\infty \int_{|X|\geqslant 1} |X|^{2(a-r)} t^{2(b-r)} dt dX \leqslant C D_r^2\end{aligned}$$

and

$$\begin{aligned}I_5 &= \int_1^\infty \int_{|X|=1/t}^1 |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dX dt \\ &\leqslant C D_r^2 \int_1^\infty \int_{|X|=1/t}^1 |X|^{2(a-r)} t^{2(b-r)} dX dt \\ &= C D_r^2 \int_1^\infty t^{2(b-r)} \int_{|X|=1/t}^1 |X|^{2(a-r)} dX dt \leqslant C' D_r^2 \int_1^\infty t^{2(b-r)}(1 - t^{-2(a-r)-1}) dt \\ &\leqslant C' D_r^2 \Big(\int_1^\infty t^{2(b-a)-1} dt + \int_1^\infty t^{2(b-r)} dt \Big) \leqslant C D_r^2.\end{aligned}$$

Finally, by using the elementary observation that $\frac{x^{2a}t^{2b}}{(1+x^r+(xt)^r)^2} \leqslant \frac{x^{2a}t^{2b}}{x^{2r}} = x^{2(a-r)}t^{2b}$ we have

$$\begin{aligned}I_6 &= \int_1^\infty \int_{1/|X|}^1 |X^a t^b \breve{w}_0(X, 2\pi p t X)|^2 dt dX \leqslant C D_r^2 \int_1^\infty \int_0^1 t^{2b} |X|^{2(a-r)} dt dX \\ &= C D_r^2 \int_1^\infty |X|^{2(a-r)} dX \int_0^1 t^{2b} dt \leqslant C' D_r^2.\end{aligned}$$

Collecting all above estimates yield the result stated in the lemma. □

We will see that, in the stable case, the position density for the linearized problem inherits these estimates in an appropriate sense.

6. Proof of Theorem 3.4.

6.1. **The Laplace transform picture.** Theorem 3.3 implies that the Laplace transforms $\tilde{n}(X,\omega)$, $\tilde{f}(X,k,\omega)$ are well-defined and analytic for $\mathrm{Re}(\omega)$ large enough. Moreover we can apply Fubini to the effect that $\tilde{n} = \mathcal{L}[\int_{\mathbb{R}^d} f dk] = \int_{\mathbb{R}^d} \mathcal{L}[f] dk$, for $\mathrm{Re}(\omega)$ large enough. The same follows for $\tilde{n}_f(X,\omega)$ by setting $P(k) = 0$.

Thus if we first take the Laplace transform of equation 6,

$$\omega\tilde{f} - f_0(X,k) - 4\pi^2 ipk\cdot X\tilde{f} + qi\Big[P\Big(k-\frac{X}{2}\Big) - P\Big(k+\frac{X}{2}\Big)\Big]\tilde{n}(X,\omega) = 0,$$

re-arrange terms

$$f(X,k,\omega) = \frac{f_0(X,k) - qi\left[P(k-\frac{X}{2}) - P(k+\frac{X}{2})\right]\tilde{n}(X,\omega)}{\omega - 4\pi^2 ipk\cdot X},$$

and integrate in k we obtain

$$\tilde{n}(X,\omega) = \int_{\mathbb{R}^d} \frac{f_0(X,k)}{\omega - 4\pi^2 ipk\cdot X} dk - qi\int_{\mathbb{R}^d} \frac{P(k-\frac{X}{2}) - P(k+\frac{X}{2})}{\omega - 4\pi^2 ipk\cdot X} dk \;\cdot\; \tilde{n}(X,\omega). \quad (36)$$

This is exactly the first expression in equation 11. From this alternative derivation we obtain that for $X \neq 0$ and $d = 1$

$$\tilde{n}_f(X,\omega) = \int_{\mathbb{R}} \frac{f_0(X,k)}{\omega - 4\pi^2 ipk\cdot X} dk = \frac{1}{4\pi ipX}\mathbb{H}[f_0(X,\cdot)]\Big(\frac{\omega}{4\pi^2 ipX}\Big) \quad (37)$$

and

$$\tilde{h}(X,\omega) = qi\int_{\mathbb{R}} \frac{P(k+\frac{X}{2}) - P(k-\frac{X}{2})}{\omega - 4\pi^2 ipk\,X} dk = \frac{q}{4\pi p}\mathbb{H}[D_X P(\cdot)]\Big(\frac{\omega}{4\pi^2 ipX}\Big). \quad (38)$$

Observation 6.1 (Case $X = 0$)**.** *For $X = 0$ we have $\tilde{h}(0,\omega) = 0$, and $\tilde{n}_f(0,\omega) = \frac{1}{\omega}\int_{\mathbb{R}} f_0(0,k)dk$, which is of course consistent with Lemma 5.1 and its consequence $\breve{n}_f(0,t) = \breve{w}_0(0,0)$. Thus it follows that $\breve{n}(0,t) = \breve{n}_f(0,t) = \breve{w}_0(0,0)$ for all t.*

Observation 6.2 (Domain of analyticity & Sokhotski-Plemelj)**.** *From the above explicit expressions it follows that, for each $X \in \mathbb{R}$, the Laplace transforms $\tilde{h}(X,\omega)$, $\tilde{n}(X,\omega)$, $\tilde{n}_f(X,\omega)$ are analytic in ω for all* $\mathrm{Re}(\omega) > 0$.

Moreover, for $X \neq 0$, we have

$$\begin{aligned}\tilde{H}(X,s) &:= \lim_{\eta\to 0}\tilde{h}(X,\eta+is)\\ &= \lim_{\eta\to 0}\frac{q}{4\pi p}\mathbb{H}[D_X P]\Big(\frac{\eta+is}{4\pi^2 ipX}\Big) = \frac{q}{4\pi p}\mathbb{S}[D_X P]\Big(\frac{s}{4\pi^2 pX}\Big)\end{aligned} \quad (39)$$

and

$$\begin{aligned}\tilde{N}_f(X,s) &:= \lim_{\eta\to 0}\tilde{n}_f(X,\eta+is)\\ &= \lim_{\eta\to 0}\frac{1}{4\pi ipX}\mathbb{H}[f_0(X,\cdot)](\frac{\eta+is}{4\pi^2 ipX}) = \frac{1}{4\pi ipX}\mathbb{S}[f_0(X,\cdot)](\frac{s}{4\pi^2 pX})\end{aligned} \quad (40)$$

by virtue of the Sokhotski-Plemelj formula, cf. Theorem C.2 in the Appendix. Moreover, observe that

$$|1-\widetilde{h}(X,\omega)|\geqslant\kappa\quad\forall X\in\mathbb{R},\,\mathrm{Re}(\omega)>0\quad\Rightarrow\quad|1-\widetilde{H}(X,s)|\geqslant\kappa\quad\forall X,s\in\mathbb{R}.\tag{41}$$

6.2. **Inverting the Laplace transform.** Recalling now equation 11, we set

$$\widetilde{I}(X,\omega):=\frac{\widetilde{h}(X,\omega)}{1-\widetilde{h}(X,\omega)}X\widetilde{n}_f(X,\omega);$$

then

$$X\check{n}(X,t)-X\check{n}_f(X,t)=\mathcal{L}^{-1}_{\omega\to t}[\widetilde{I}(X,\omega)].$$

Observe also that equations 39 and 40 imply

$$I(X,s):=\lim_{\eta\to0^+}\widetilde{I}(X,\eta+is)=\frac{\widetilde{H}(X,s)}{1-\widetilde{H}(X,s)}X\widetilde{N}_f(X,s).$$

We will use Theorem C.3 from the Appendix to compute $\mathcal{L}^{-1}_{\omega\to t}[\widetilde{I}(X,\omega)]$ for each $0\neq X\in\mathbb{R}$. To that end, we will need to check that its assumptions are satisfied, namely that $\widetilde{I}(X,\omega)$ is bounded and analytic on $\{\mathrm{Re}(\omega>0\})$; that $|\widetilde{I}(X,\omega)|$ decays uniformly as $|\omega|\to\infty$; and that $I(X,\cdot)\in L^1(\mathbb{R})\cap C^0(\mathbb{R})$ for all $X\in\mathbb{R}$;

First of all, Observation 6.2 directly implies that $\widetilde{I}(X,\omega)$ is bounded and analytic on the open half-plane $\{\mathrm{Re}(\omega>0\})$. Moreover,

$$\lim_{\rho\to\infty}\sup_{\substack{\mathrm{Re}(\omega)>0\\|\omega|>\rho}}|\widetilde{I}(X,\omega)|\leqslant\sup_{\mathrm{Re}(\omega')>0}|X\widetilde{n}_f(X,\omega')|\cdot\lim_{\rho\to\infty}\sup_{\substack{\mathrm{Re}(\omega)>0\\|\omega|>\rho}}|\widetilde{h}(X,\omega)|=0$$

where in the last step we used Lemma A.2 from the Appendix.

Finally, the expression for $I(X,s)$ implies that it is continuous in s. To show that $I(X,\cdot)\in L^1(\mathbb{R})$ uniformly in X observe that

$$\int_{\mathbb{R}}|I(X,s)|ds\leqslant\frac{1}{\kappa}\sup_{s'\in\mathbb{R}}|X\widetilde{N}_f(X,s')|\int_{\mathbb{R}}|\widetilde{H}(X,s)|ds\leqslant C,$$

where we used property 41, Lemma 5.2 for $X\widetilde{N}_f$, and Theorem A.3 for $\widetilde{H}$ (observe in particular that, by construction, D_XP is a function of compact support with integral $\int_{\mathbb{R}}D_XP(k)dk=0$ for all $0\neq X\in\mathbb{R}$, hence Theorem A.3 indeed applies).

So all the assumptions of Theorem C.3 are satisfied, and we can apply it to the effect that

$$X\check{n}(X,t)-X\check{n}_f(X,t)=\int_{-\infty}^{\infty}e^{i\,s\,t}\frac{\widetilde{H}(X,s)}{1-\widetilde{H}(X,s)}X\widetilde{N}_f(X,s)ds.\tag{42}$$

Remark 6.1. If one tries to use Theorem C.3 directly on $X\widetilde{n}(X,\omega)$ then the only way to guarantee the L^1-property required in equation 66 seems to be requiring $\int_{k\in\mathbb{R}}f_0(X,k)dk=0$ for all $X\in\mathbb{R}$. Here instead we only require that the limit as $\eta\to0^+$ of the difference $X\widetilde{n}(X,s+i\eta)-X\widetilde{n}_f(X,s+i\eta)$ is in $L^1_s(\mathbb{R})$ uniformly in X, avoiding any extra assumptions on the initial data.

6.3. **Space-time estimates for the force.** First we use equation 42 to prove the following estimate

Lemma 6.3. *Let $a>1$, $b>0$, and moreover recall that, since $f_0\in\mathcal{S}(\mathbb{R}^2)$,*

$$|\check{w}_0(X,K)|\leqslant\frac{\|w_0\|_{\Sigma^{r',\infty}}}{1+|X|^r+|K|^r}$$

for any r, in particular for $r > \max\{a+\frac{1}{2}, b+\frac{1}{2}\}$. Then there exists a $C = C(a,b,P)$ so that

$$\|t|X|^a \breve{n}\|_{L^2_{X,t}} + \||X|^b \breve{n}\|_{L^2_{X,t}} \leqslant \frac{C(a,b,P) D_r}{\kappa}$$

for all $0 \neq X \in \mathbb{R}$. Note that, by virtue of Observation 5.3, $D_r \leqslant C\|w_0\|_{\Sigma^{r',\infty}}$ for some r' sufficiently large.

Proof. First we will bound $X^a t^b \breve{n}$ norms from appropriate quantities involving $\breve{n}_f$. Using the alternate Fourier transform $\mathfrak{F}$, introduced in Section 2.1, on 42 it follows that

$$X\breve{n}(X,t) - X\breve{n}_f(X,t) = \mathfrak{F}^{-1}_{s\to t}\left[\frac{X\tilde{N}_f(X,s)\tilde{H}(X,s)}{1-\tilde{H}(X,s)}\right].$$

Thus

$$\begin{aligned}\|X^b\breve{n} - X^b\breve{n}_f\|_{L^2_{X,t}} &= \left\|\mathfrak{F}^{-1}_{s\to t}\left[\frac{X^b\tilde{N}_f(X,s)\tilde{H}(X,s)}{1-\tilde{H}(X,s)}\right]\right\|_{L^2_{X,t}} \\ &= C\left\|\frac{X^b\tilde{N}_f(X,s)\tilde{H}(X,s)}{1-\tilde{H}(X,s)}\right\|_{L^2_{X,s}} \\ &\leqslant C\left(\sup_{X,s}\left|\frac{1}{1-\tilde{H}(X,s)}\right|\right)\left(\sup_{X,s}|\tilde{H}(X,s)|\right)\|X^b\tilde{N}_f\|_{L^2_{X,s}}.\end{aligned}$$

For the first factor we use equation 41. For the second factor observe that, by virtue of Theorem C.1, we have

$$\begin{aligned}\sup_{X,s}|\tilde{H}(X,s)| &= |\frac{q}{4\pi p}|\sup_{\zeta,t}|\mathbb{S}[D_\zeta P](t)| \\ &\leqslant C\sup_{\zeta}\|\mathbb{S}[D_\zeta P]\|_{H^1} \leqslant C'\sup_{\zeta}\|D_\zeta P\|_{H^1} \leqslant C'',\end{aligned} \tag{43}$$

so finally

$$\|X^b\breve{n}\|_{L^2_{X,t}} \leqslant C\|X^b\tilde{N}_f\|_{L^2_{X,s}} = C'\|X^b\breve{n}_f\|_{L^2_{X,t}} \tag{44}$$

since $\breve{n}_f = \mathfrak{F}^{-1}[\tilde{N}_f]$. Now working similarly and using equation 42 we have

$$X^a t\big[\breve{n}(X,t) - \breve{n}_f(X,t)\big] = i\int_{-\infty}^{\infty} e^{ist}\partial_s\frac{X^a\tilde{N}_f(X,s)\tilde{H}(X,s)}{1-\tilde{H}(X,s)}ds,$$

which implies

$$\begin{aligned}\|X^a t(\breve{n}(X,t) - \breve{n}_f(X,t))\|_{L^2_{X,t}} &= C\left\|\partial_s\frac{X^a\tilde{N}_f(X,s)\tilde{H}(X,s)}{1-\tilde{H}(X,s)}\right\|_{L^2_{X,s}} \\ &\leqslant C\|\partial_s X^a\tilde{N}_f\|_{L^2_{X,s}}\sup_{X,s}\left|\frac{\tilde{H}(X,s)}{1-\tilde{H}(X,s)}\right| \\ &\quad + C\left(\int_{\mathbb{R}^2}|X^a\tilde{N}_f(X,s)|^2\Big[\sup_{s'}\left|\frac{\partial_s\tilde{H}(X,s')}{(1-\tilde{H}(X,s'))^2}\right|\Big]^2 dsdX\right)^{\frac{1}{2}}.\end{aligned} \tag{45}$$

Now, observe that

$$\|\partial_s X^a\tilde{N}_f\|_{L^2_{X,s}} = \|tX^a\breve{n}_f\|_{L^2_{X,t}}$$

by virtue of a Fourier transform; $|\widetilde{H}(X,s)/(1-\widetilde{H}(X,s))| \leqslant C'''/\kappa^2$ by virtue of 41 and 43; and

$$\begin{aligned}\Big|\frac{\partial_s\widetilde{H}(X,s)}{(1-\widetilde{H}(X,s))^2}\Big| &\leqslant \frac{C}{\kappa^2}\Big|\partial_s\widetilde{H}(X,s)\Big| \\ &= \frac{C'}{\kappa^2}\Big|\partial_s\mathbb{S}[D_XP]\Big(\frac{s}{4\pi^2pX}\Big)\Big| = \frac{C''}{\kappa^2}\frac{1}{|X|}\Big|\mathbb{S}[D_XP']\Big(\frac{s}{4\pi^2pX}\Big)\Big|\end{aligned} \tag{46}$$

so that by collecting all this and inserting it back in 45 we get

$$\begin{aligned}\|X^at(\check{n}(X,t)-\check{n}_f(X,t))\|_{L^2_{X,t}} \leqslant \frac{C}{\kappa}\|tX^a\check{n}_f\|_{L^2_{X,t}} \\ + \frac{C}{\kappa^2}\sup_{s'}\Big|\mathbb{S}[D_XP'](\frac{s'}{4\pi^2pX})\Big|\|X^{a-1}\check{n}_f\|_{L^2_{X,t}}.\end{aligned}$$

Using our assumptions on P we have

$$\sup_{s'\in\mathbb{R}}|\mathbb{S}[D_XP']\Big(\frac{s'}{4\pi^2pX}\Big)| \leqslant \sup_{\zeta,\tau}|\mathbb{S}[D_\zeta P'](\tau)| \leqslant \sup_{\zeta\in\mathbb{R}}\|D_\zeta P'\|_{H^1(\mathbb{R})} \leqslant C,$$

and therefore

$$\|tX^a\check{n}\|_{L^2_{X,t}} \leqslant C\left(\frac{1}{\kappa}+\frac{1}{\kappa^2}\right)\left(\|tX^a\check{n}_f\|_{L^2_{X,t}}+\|X^{a-1}\check{n}_f\|_{L^2_{X,t}}\right) \tag{47}$$

Then the result of the lemma follows by combining estimates 44 and 47 together with Lemma 5.4. □

Applying now Lemma 6.3 we obtain estimate 18 stated in Theorem 3.4.

6.4. **Construction of the wave operator.** Equation 7 implies

$$\begin{aligned}e^{-4\pi^2ipk\cdot Xt}f(X,k,t)-f_0(X,k) &= J(X,k,t), \\ J(X,k,t) :=& qi\int_0^t e^{-4\pi^2ipk\cdot X\tau}\frac{P\left(k+\frac{X}{2}\right)-P\left(k-\frac{X}{2}\right)}{X}X\check{n}(X,\tau)d\tau\end{aligned} \tag{48}$$

For any $0<\theta<1/2$ and $\gamma>1$ using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_{\mathbb{R}}|J(k,X,t)|dX &\leqslant C\sup_{\zeta,s}|D_\zeta P(s)|\int_{\mathbb{R}}\int_0^t|X\check{n}(X,\tau)|d\tau dX \\ &\leqslant C'\int_0^{+\infty}\int_0^t\frac{\sqrt{1+(|X|^\theta\tau)^2+|X|^{2\gamma}}}{\sqrt{1+(|X|^\theta\tau)^2+|X|^{2\gamma}}}|X\check{n}(X,\tau)|d\tau dX \\ &\leqslant C''\sqrt{\int_{\mathbb{R}}\int_0^t\Big[1+(|X|^\theta\tau)^2+|X|^{2\gamma}\Big]|X|^2|\check{n}(X,\tau)|^2d\tau dX} \\ &\quad\times\sqrt{\int_0^{+\infty}\int_0^t\frac{1}{1+(|X|^\theta\tau)^2+|X|^{2\gamma}}d\tau dX}.\end{aligned}$$

The first factor in the last estimates is estimated by

$$\begin{aligned}&\sqrt{\int_0^{+\infty}\int_0^t\Big[1+(|X|^\theta\tau)^2+|X|^{2\gamma}\Big]|X\check{n}(X,\tau)|^2d\tau dX} \leqslant \|(1+|X|^\theta t+|X|^\gamma)X\check{n}\|_{L^2_{X,t}} \\ &\leqslant \|X\check{n}\|_{L^2_{X,t}}+\|t|X|^{1+\theta}\check{n}\|_{L^2_{X,t}}+\||X|^{1+\gamma}\check{n}\|_{L^2_{X,t}} \leqslant C(\theta,\gamma,P)\|w_0\|_{\Sigma^{r',\infty}}\end{aligned}$$

for some r' large enough by virtue of Lemma 6.3. For the other factor we break the integral up over the contributions from different regions,

$$\int_0^{+\infty}\int_0^t \frac{1}{1+(|X|^\theta\tau)^2+|X|^{2\gamma}}d\tau dX = I_1+I_2+I_3+I_4+I_5+I_6,$$

where we use the same breakdown as in Figure 1. Without loss of generality we assume $t>1$. Then the first integral is estimated as

$$I_1 \leqslant \int_0^1\int_0^1 d\tau dX = 1.$$

For the second integral we have

$$\begin{aligned} I_2 \leqslant& \int_1^\infty\int_0^{1/t}\frac{1}{x^{2\theta}\tau^2}d\tau dx = \int_1^\infty \tau^{-2}\int_{x=0}^{1/t} x^{-2\theta}d\tau dx \\ =& C\int_1^\infty \tau^{-2+2\theta-1}d\tau = C(1+t^{-2+2\theta}) \leqslant C'. \end{aligned}$$

Here we used $-2\theta>-1 \iff \theta<1/2$ for the integral with respect to x to exist and $-3+2\theta<-1 \iff \theta<1$ for the integral with respect to τ to exist. Moreover

$$I_3 \leqslant \int_1^\infty\int_0^{1/x} x^{-2\gamma}dtdx = \int_1^\infty x^{-2\gamma-1}dx = C,$$

where we used $-2\gamma-1<-1 \iff \gamma>0$. For I_4 we refer to Lemma A.1 in the Appendix, where setting $\zeta=3/4$ leads to

$$\frac{1}{(x^\theta\tau)^2+x^{2\gamma}} \leqslant \frac{1}{(x^\theta\tau)^{\frac{3}{2}}x^{\frac{\gamma}{2}}} = \tau^{-\frac{3}{2}}x^{-\frac{\gamma}{2}-\frac{3}{2}\theta}.$$

Thus

$$I_4 \leqslant C\int_1^\infty\int_1^t \tau^{-\frac{3}{2}}x^{-\frac{\gamma}{2}-\frac{3}{2}\theta}d\tau dx = C\left(\tau^{-\frac{1}{2}}\Big|_1^t\right)\left(x^{1-\frac{\gamma}{2}-\frac{3}{2}\theta}\Big|_1^\infty\right) = C'(1+t^{-\frac{1}{2}}),$$

where we used the fact that, by assumption, $\gamma/2+3\theta/2>5/4>1$. The next integral is estimated as

$$\begin{aligned} I_5 \leqslant& C\int_0^1\int_{1/x}^\infty \frac{1}{x^{2\theta}t^2}dtdx = C\int_0^1 x^{-2\theta}\int_{1/x}^\infty \tau^{-2}d\tau dx \\ =& C\int_0^1 x^{-2\theta}\left(\tau^{-1}\Big|_{1/x}^\infty\right)dx = C\int_0^1 x^{1-2\theta}dx = C', \end{aligned}$$

since $1-2\theta>-1 \iff \theta<1/2$. Finally,

$$I_6 \leqslant C\int_1^\infty\int_{1/x}^1 x^{-2\gamma}d\tau dx \leqslant \int_1^\infty x^{-2\gamma-1}dx \leqslant C.$$

So we showed that

$$\int_{\mathbb{R}}|J(k,X,t)|dX \leqslant C\|w_0\|_{\Sigma^{r',\infty}}.$$

Since $J(X,k,t)$ is an absolutely convergent integral in t, the uniform-in-t bound automatically implies the existence of

$$J^{\infty}(X,k) := \lim_{t\to\infty} J(X,k,t)$$
$$= qi\int_0^{\infty} e^{-4\pi^2 ipk\cdot X\tau}\frac{P\left(k+\frac{X}{2}\right)-P\left(k-\frac{X}{2}\right)}{X}X\check{n}(X,\tau)d\tau \in L^{\infty}(\mathbb{R};L^1(\mathbb{R})).$$

Now equation 48 can be recast as

$$U(-t)f(t)-f_0 = J(t) \qquad \Rightarrow \qquad \lim_{t\to\infty}\big(U(-t)f(t)-f_0\big) = J^{\infty}.$$

By setting $\mathbb{W}(w_0) := \mathcal{F}_{X\to x}[f_0+J^{\infty}]$, we have

$$\begin{aligned}\|w(t)-E(t)\mathbb{W}(w_0)\|_{L^{\infty}(\mathbb{R}^2)} &\leqslant \|f(t)-U(t)\big(f_0+J^{\infty}\big)\|_{L^{\infty}(\mathbb{R};L^1(\mathbb{R}))}\\ &= \|U(-t)f(t)-f_0-J^{\infty}\|_{L^{\infty}(\mathbb{R};L^1(\mathbb{R}))},\end{aligned}$$

hence equation 19 follows.

Remark 6.2. Observe that by collecting the above it follows that

$$\|\mathbb{W}(w_0)\|_{L^{\infty}_{x,k}} \leqslant \|w_0\|_{L^{\infty}_{x,k}} + \|J^{\infty}\|_{L^{\infty}_k L^1_X} \leqslant C'\|w_0\|_{\Sigma^{r',\infty}}.$$

7. **Proof of Theorem 3.5.** In this section we present the proof of the last main theorem. We split the proof into four parts.

7.1. **Elaboration and symmetry of (A).** Assuming condition (A) holds, there exists a sequence $(X_n,\omega_n) = (X_n, a_n+ib_n) \in \mathbb{R}\times\{\mathrm{Re}(z)>0\}$ such that

$$\lim_{n\to\infty}\widetilde{h}(X_n,\omega_n) = 1.$$

Without loss of generality we can assume $X_n \neq 0$ for all $n\in\mathbb{N}$ (it suffices to observe that $\widetilde{h}(0,\omega)=0$ for all ω). Note that X_* can still be zero.
Symmetry: The expression for $\tilde{h}(X,\omega)$ in 38, namely

$$\widetilde{h}(X,\omega) = qi\int_{\mathbb{R}}\frac{P(k+\frac{X}{2})-P(k-\frac{X}{2})}{\omega-4\pi^2 ipk\,X}dk,$$

yields that, for $X_n, a_n, b_n \in \mathbb{R}$ as above, we have the following equivalence

$$\lim_{n\to\infty}\widetilde{h}(X_n,a_n+ib_n)=1 \iff \lim_{n\to\infty}\widetilde{h}(X_n,-a_n+ib_n)=1,$$

i.e.

$$\exists X_n\in\mathbb{R},\ \omega_n\in\mathbb{C}\ :\ \widetilde{h}(X_n,\omega_n)\to 1 \iff \exists X_n\in\mathbb{R},\ \mathrm{Re}(\omega_n)\geqslant 0\ :\ \widetilde{h}(X_n,\omega_n)\to 1.$$

Indeed all the conditions $(A),(B)$ and (C) have this symmetry.
Claim I: *The sequence (X_n,ω_n) is bounded.*

Proof. If $|X_n|+|\omega_n|\to\infty$, then

$$\lim_{n\to\infty}\widetilde{h}(X_n,\omega_n) = qi\lim_{n\to\infty}\int_{\mathbb{R}}\frac{P(k+\frac{X_n}{2})-P(k-\frac{X_n}{2})}{\omega_n-4\pi^2 ipk\,X_n}dk = 0\neq 1.$$

□

Thus (X_n, ω_n) *has accumulation points in* $\mathbb{R} \times \{\mathrm{Re}(z) \geqslant 0\}$ *and from now on we will denote*

$$(X_*, a_* + ib_*) = (X_*, \omega_*) := \lim_{n\to\infty} (X_n, \omega_n), \tag{49}$$

up to extraction of a subsequence.

Claim II: *Denote*

$$\Omega_n := \frac{\omega_n}{4\pi p i X_n} = \frac{b_n - ia_n}{4\pi p X_n}. \tag{50}$$

Then Ω_n *is bounded.*

Proof. First of all observe that Ω_n is well-defined since, as we saw above, $X_n \neq 0$. By virtue of equation 38,

$$\widetilde{h}(X_n, \omega_n) = \frac{q}{4\pi p} \mathbb{H}[D_{X_n} P](\Omega_n). \tag{51}$$

Clearly, if $|\Omega_n| \to \infty$ then $(q/4\pi p)\mathbb{H}[D_{X_n} P](\Omega_n) \to 0 \neq 1$. Thus, by extracting yet another subsequence if necessary, we have $(X_n, \Omega_n) \to (X_*, \Omega_*) \in \mathbb{R} \times \mathbb{C}$. □

7.2. **Proof of** $(\boldsymbol{A}) \iff (\boldsymbol{B})$**.** We will examine separately the following two cases:

Case 1: If $\mathrm{Im}(\Omega_*) \neq 0$ then, by continuity,

$$\widetilde{h}(X_n, \omega_n) \to 1 \iff \frac{q}{4\pi p} \mathbb{H}[D_{X_*} P](\Omega_*) = 1.$$

Case 2: If $\mathrm{Im}(\Omega_*) = 0$ then, by the Sokhotski-Plemelj formula (cf. Theorem C.2), for $X_* > 0$ we have

$$\widetilde{h}(X_n, \omega_n) \to 1 \iff \frac{q}{4\pi p} \mathbb{S}[D_{X_*} P](\Omega_*) = 1 \iff \left\{ \begin{array}{c} \frac{q}{4\pi p} \mathbb{H}[D_{X_*} P](\Omega_*) = 1, \\ i\frac{q}{4\pi p} D_{X_*} P(\Omega_*) = 0, \end{array} \right\}$$

while for $X_* < 0$ we have $\overline{\mathbb{S}[D_{X_*} P](\Omega_*)} = 1$, leading to the same end result. For $X_* = 0$ observe that both one-sided limits $\mathrm{Im}(\Omega_n) \to 0^{\pm}$, yield the same result as well.

Checking that (B) implies (A) is obvious.

7.3. **Proof of** $(\boldsymbol{B}) \iff (\boldsymbol{C})$**.** Denote $\mathbb{F}_X(\Omega) := \mathbb{H}[D_X P](\Omega)$. Like before, if $\pm X_* > 0$ we have $\pm \mathrm{Im}(\Omega_*) < 0$, and for $X_* = 0$ we should take each one-sided limit separately. All these cases follow the same steps, so without loss of generality we only present the case $X_* > 0$.

Assume Case 1 of (B) above holds, i.e. $\exists X_* > 0, \mathrm{Im}(\Omega_*) \neq 0$ such that $\mathbb{H}[D_{X_*} P](\Omega_*) = 4\pi p/q$.

Then by virtue of the argument principle [26], for any contour γ within the lower half-plane containing Ω_*, its image $\mathbb{F}_{X_*}(\gamma) := \{z | \exists w \in \gamma : z = \mathbb{F}_X(w)\}$ is enclosing $4\pi p/q$. Let us select γ_η the closed contour comprised by parts of the horizontal line $\mathbb{R} - i\eta$ and the semicircle $\{\frac{e^{i\theta}}{\eta}, \quad \theta \in (-\pi, 0)\}$. Clearly, Ω_* will eventually be enclosed by γ_η for η small enough, thus $\mathbb{F}_{X_*}(\gamma_\eta)$ is enclosing $4\pi p/q$ for η small enough. Using the decay properties of $\mathbb{F}_{X_*}(\omega)$ as $|\omega| \to \infty$ (cf. Lemma A.2 in the Appendix) and the Sokhotski-Plemelj formula, it follows that $\lim_{\eta\to 0} \mathbb{F}_{X_*}(\gamma_\eta) = \Gamma_X$ as defined in equation 20, i.e. $4\pi p/q \in \overset{\circ}{\Gamma}_{X_*}$.

If Case 2 of (B) above holds, denote Ω_n a sequence of points on Γ_{η_n} such that $\Omega_n \to \Omega_*$; then by construction $\lim_{n\to\infty} \mathbb{F}_{X_*}(\Omega_n) = 4\pi p/q$ and therefore $4\pi p/q \in \lim_{\eta\to 0} \mathbb{F}_{X_*}(\gamma_\eta) = \Gamma_{X_*}$.

To prove that $(C) \implies (B)$, first we need to observe that, since $\lim_{|X|\to\infty} \|D_X P\|_{H^1} = 0$, there exists $M > 0$ such that for $|X| > M$ all points of Γ_X are inside $\{z \in \mathbb{C} \,|\, |z| < 2\pi p/q\}$. Thus $4\pi p/q \in \overline{\Gamma}$ implies $\exists X_* \in [-M, M]$ such that $d(4\pi p/q, \overset{\circ}{\Gamma}_{X_*})$. One now readily checks that there exists Ω_* with $\mathrm{Im}(\Omega_*) \leqslant 0$ such that $\lim_{\substack{\Omega\to\Omega_* \\ \mathrm{Im}(\Omega)<0}} \mathbb{F}_{X_*}(\Omega) = 4\pi p/q$.

7.4. **Sufficient condition for stability.** This follows from the elementary observation that, for the curve Γ_X on the complex plane, which starts and ends at 0, to be winding around the real number $4\pi p/q$, it is necessary to intersect the real axis somewhere on the right of $4\pi p/q$. The argument can be easily adapted for the limiting case $4\pi p/q \in \Gamma_X$. See also Figures 3, 4 for a visualization of this point.

The proof is completed by observing that, according to equation 20, $\Gamma_X = \{\mathbb{S}[D_X P](t), t \in \mathbb{R}\}$ intersects the real axis only for those t_* that are quasi-critical points, $D_X P(t_*) = 0$.

8. Applications.

8.1. **The question: Are realistic sea states modulationally (un)stable?** Landau damping for the Alber equation (i.e. dispersion of inhomogeneities in the presence of a homogeneous background) has been conjectured at least since [23], but no precise results existed before the one presented here. In this paper we establish rigorously the decay of inhomogeneities in the stable case, but for ocean engineers the most immediate question is a practical and reliable way to investigate whether a given spectrum is stable or not.

Alber's "eigenvalue relation" is a system of two (real valued) nonlinear equations in three (real) unknowns, which in general has one-dimensional manifolds of solutions. Determining whether such a system has solutions or not is not straightforward, and has attracted a lot of attention in the ocean waves community [13, 23, 29, 32]. In [13] a state-of-the-art investigation of this question is presented, describing the challenges. We will show that criterion (C) of Theorem 3.5 provides a reliable and more straightforward way to investigate the modulational stability of any given spectrum. But first let us go over how we choose the spectra to be investigated.

8.2. **JONSWAP spectra and the North Atlantic Scatter Diagram.** While the power spectrum of a sea state can in principle be directly measured, in practice often parametric spectra are used. A widely used such parametric spectrum is the so-called JONSWAP spectrum (the initials stand for "Joint North Sea Wave Project", and some typical profiles can be found in Figure 2),

$$\begin{aligned} S_{\alpha,\gamma,k_0}(k) = S(k) =& \frac{\alpha}{2k^3} e^{-\frac{5}{4}(\frac{k_0}{k})^2} \gamma^{exp[-(1-\sqrt{k/k_0})^2/2\delta^2]}, \\ \delta =& \delta(k) = \begin{cases} 0.07, & k \leqslant k_0, \\ 0.09, & k > k_0. \end{cases} \end{aligned} \tag{52}$$

This was introduced in [16] following extensive study of measured nonparametric spectra, and it incorporates several physical insights: it is effectively zero in a

neighbourhood of $k = 0$, it has a power-law decay for $k \gg 1$, and it is unimodal. The free parameters are $\alpha > 0$, which increases with the power of the sea state (i.e. larger α leads to larger significant wave height H_s), $\gamma > 1$ which increases with the "peakiness" of the spectrum (i.e. larger γ leads to more peaked spectra, with larger H_s as well), and k_0 stands for the peak wavenumber. Very often a JONSWAP spectrum is fitted to a time-series of point measurements for the frequency ω (instead of the wavenumber k) but, assuming unidirectional propagation, the conversion between a wavenumber-resolved and frequency-resolved spectra is standard [22]. It

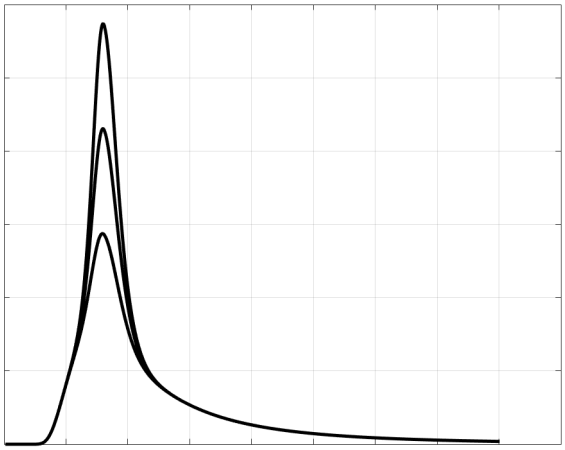

FIGURE 2. Some common profiles of JONSWAP spectra.

is widely used in the study of realistic sea states, e.g. [13, 29, 10, 39], as it offers an intuitive and plausible parametrization of spectra in terms of power, peakiness and carrier wavenumber.

Now the question becomes, what are some realistic values for α, γ and k_0 corresponding to various plausible scenarios in the ocean? A canonical data set has been created precisely in this context; it is called the North Atlantic Scatter Diagram [39, p. 244], and it includes measured statistics from 100000 sea states in the North Atlantic, along with the likelihood for each sea state. A JONSWAP spectrum (i.e. α, γ and k_0 values) can then be fitted to each sea state using state of the art engineering practice [39, Section 3.5.5]. The fact that parameter values are fitted and not measured directly has a few implications: for example, several sea states end up having $\gamma = 1$ (smallest allowed value) or $\gamma = 5$ (largest allowed value). More importantly, it is a priori possible that we could end up with some modulationally unstable spectra through this route. In contrast, if power spectra were measured directly, it doesn't seem likely that an "unstable spectrum" could be robustly measured at all.

So now it should be clear how we choose the spectra to investigate: we will work with JONSWAP spectra, fitted to the North Atlantic Scatter Diagram according to the state of the art [39]. Ultimately each blue star in Figure 5 corresponds to one such JONSWAP spectrum, and it has a known likelihood of being observed at a random point in the North Atlantic, at a random time of the year (this likelihood is not plotted here, but can be found in [39]).

8.3. **Implementation.** One should start with the important observation that, for the question of modulation instability of JONSWAP spectra, k_0 happens to play no role[2]. This is well-known [13, 29], but we will demonstrate it for completeness.

Let us begin with Alber's eigenvalue relation for some JONSWAP spectrum $S(k)$, e.g. as in equation (2) of [13]:

$$\exists \Omega \in \mathbb{C},\ X \in \mathbb{R} \qquad 1 + \omega_0 k_0^2 \int\limits_{k\in\mathbb{R}} \frac{S(k+\frac{X}{2}) - S(k-\frac{X}{2})}{\Omega + \frac{\omega_0}{4k_0^2} kX} dk = 0 \tag{53}$$

Recall that the existence of such Ω, X means the spectrum is unstable. By rescaling $X' = X/k_0$, $\Omega' := -\Omega 4k_0/(X\omega_0)$, and changing variable $k' = k/k_0$ problem 53 is seen to be equivalent to

$$\exists \Omega' \in \mathbb{C},\ X' \in \mathbb{R} \qquad \mathbb{H}[D_{X'}P](\Omega') = \frac{1}{4\pi} \tag{54}$$

where

$$P(k) = \frac{\alpha}{2k^3} e^{-\frac{5}{4}k^{-2}} \gamma^{exp[-(1-\sqrt{k})^2/2\delta^2]}, \qquad \delta = \delta(k) = \begin{cases} 0.07, & k \leqslant 1, \\ 0.09, & k > 1, \end{cases} \tag{55}$$

is the JONSWAP spectrum with $k_0 = 1$ and the original α, γ. So the original value of k_0 will play no further role in checking stability.

To actually do the checking, we recall part (C) of Theorem 3.5: instability exists if and only if

$$\exists X \in \mathbb{R} \ :\ \frac{1}{4\pi} \text{ is on, or enclosed by, the curve } \Gamma_X := \lim_{\eta\to 0^+} \mathbb{H}[D_X P](t - i\eta). \tag{56}$$

So instead of checking for the existence of solutions of a system of nonlinear equations, we simply check whether $1/4\pi$ is on, or enclosed by, a curve in the complex plane. The computation of the curve itself is somewhat demanding, since it involves a very nearly singular integral. Still, it can be done much more reliably and quickly than checking for existence of solutions of 53.

After some numerical testing, it is found sufficient to approximate

$$\Gamma_X(t) = \lim_{\eta\to 0} \mathbb{H}[D_X P(\cdot)](t - i\eta) \approx \mathbb{H}[D_X P(\cdot)](t - i\texttt{tol}), \qquad \texttt{tol=1e-4}.$$

In all relevant cases here we observe that condition 56 is satisfied if and only if it is satisfied for $X = 0$ (and this seems to be the case for any unimodal spectrum). Once we generate an approximation to Γ_X, the built-in MATLAB function `inpolygon` is then used to determine if the target $4\pi p/q$ is contained in $\Gamma_X \cup \{0\}$.

Application to individual spectra is vizualized in Figures 3 and 4. Synoptic plots showing the stable and unstable regions of the $\gamma - \alpha$ plane can be found in Figure 5. There is broad agreement with [13, 29], but we find somewhat fewer unstable sea states. Modulationally unstable sea states are the prime suspects for rogue waves [8, 4, 10, 11, 14, 25], and we find that such sea states are very unlikely but nevertheless they do exist, with an estimated total likelihood of $\approx 2 \cdot 10^{-3}$. This is broadly consistent with the record of observations of rogue waves.

[2]We would like to thank A. Babanin and O. Gramstad for their helpful insights on this point.

8.4. **The bifurcation from Landau damping to modulation instability.** Another aspect of practical interest is to understand the bifurcation from stability to instability e.g. as α or γ increases. This has been thought of as a violent change in behavior once a borderline stable spectrum became unstable. Such a change in behavior is the object of numerical experiments in [17], where it is noted that instead only a gradual transition is found. In fact, the lack of a dramatic bifurcation was seen as a challenge for the validity Alber equation in the aforementioned works. However our proof here (and the heuristic results of [4] for the unstable case) show that indeed the Alber equation only predicts a gradual transition.

For example, assume γ_*, α_* are exactly on the separatrix of the stable/unstable regions as in Figure 5. Also take (γ_m, α_m) a sequence of points in the stable region with $\lim_{m\to\infty}(\gamma_m, \alpha_m) = (\gamma_*, \alpha_*)$. Now denote $S_*(k) = S_{\alpha_*,\gamma_*,k_0}(k)$, $S_m(k) = S_{\alpha_m,\gamma_m,k_0}(k)$. For each $S_m(k)$ we have Landau damping, and dispersion of inhomogeneities over a timescale controlled by κ_m. However, as $m \to \infty$ it takes longer and longer for the inhomogeneities to disperse; this can be seen e.g. by considering equation 11, which in this case becomes

$$X\widetilde{n}_m(X,\omega) - X\widetilde{n}_f(X,\omega) = \frac{\widetilde{h}_m(X,\omega)}{1-\widetilde{h}_m(X,\omega)} X\widetilde{n}_f(X,\omega),$$

assuming the same initial inhomogeneity for all m. So when $m \to \infty$ we have $\kappa_m \to 0$ and the force decays more and more slowly, until it ceases to have any time decay at all.

On the other hand, in the unstable case a very slow rate of growth would make the instability irrelevant; moreover, a very small bandwidth of unstable wavenumbers X would make the resulting extreme events supported over unrealistically large regions (e.g. thousands of wavelengths) [4]; but there are no energy transport mechanisms to support such events. In other words, to really observe the modulation instability a fast enough rate of growth and a large enough bandwidth of unstable wavenumbers are required.

So a barely stable and a barely unstable spectrum would lead to very similar behaviour over physically relevant timescales and lengthscales, reconciling the findings of [17] with the analysis of the Alber equation.

8.5. **1 versus 2 spatial dimensions.** It must be noted that in the original paper [1] a two-dimensional setup is used, with the Davey-Stewartson equation for the envelope instead of the NLS equation 2. However, while technically two-dimensional, the Davey-Stewartson equation has unidirectional propagation built in, and the second dimension is merely the "transverse direction". This leads to Alber's "eigenvalue relation" eventually being one-dimensional: an effective spectrum is used, that results from appropriate integration of the two-dimensional spectrum along the transverse direction. In that sense, Theorem 3.5 can be used in $1+1$ dimensional scenarios automatically, as the effective stability condition is one-dimensional anyway and of the exact same form as the one treated here.

In genuinely two-dimensional settings (e.g. crossing seas), things are more complicated: the NLS equation 2 is no longer an appropriate model. Systems of NLS equations [24, 30, 31] or systems of other dispersive equations [14] have been proposed. In any case the departure point is no longer a single scalar NLS equation.

8.6. **Other problems.** More broadly, it must also be mentioned that combining NLS-type equations with stochastic modelling is natural in many different contexts,

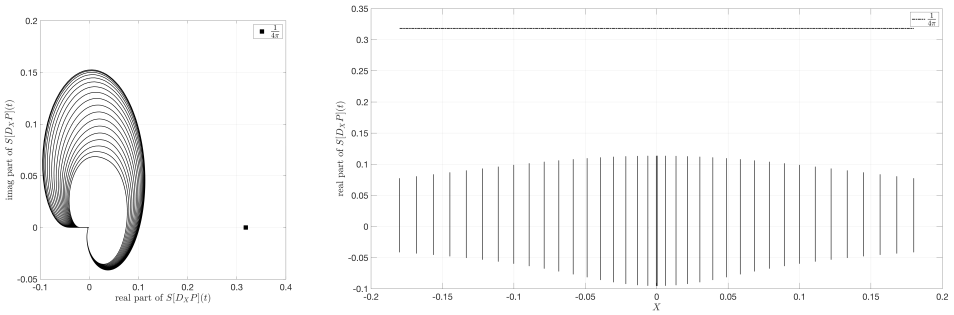

FIGURE 3. Numerical investigation of the stability condition for a stable JONSWAP spectrum, cf. Section 8 for more details. We are using a target of $1/4\pi$ as in equation 54. **Left:** Plots of the curve Γ_X on the complex plane for different values of X. Since $1/4\pi$ is always outside the Γ_X, this spectrum is stable. **Right:** The span of the real parts of Γ_X for different values of X.

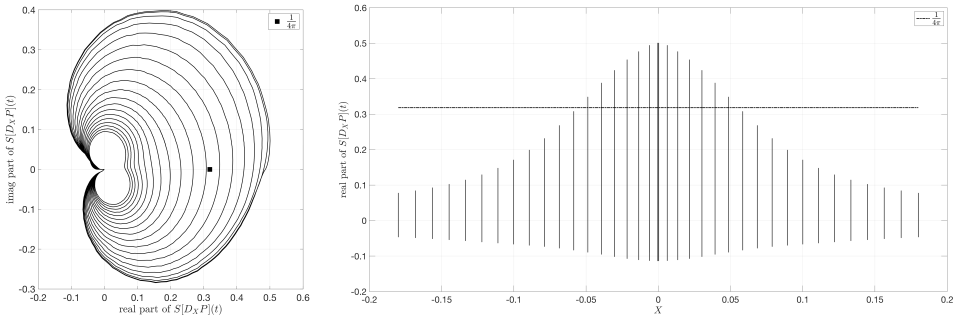

FIGURE 4. Numerical investigation of the stability condition for an unstable JONSWAP spectrum. **Left:** Plots of the curve Γ_X on the complex plane for different values of X. Since $1/4\pi$ is contained in some curves Γ_X, the spectrum is unstable. **Right:** The span of the real parts of Γ_X for different values of X. In this case it highlights clearly the bandwidth of unstable wavenumbers X.

not only ocean waves. It is thus natural that variants of the Alber equation are being independently rederived in different branches of physics, including optics [15] and many-particle systems [12]. Thus the main results of this paper are, in principle, applicable and/or generalizable to other problems as well.

Acknowledgments. We would like to thank C. Saffirio, O. Gramstad and A. Babanin for helpful discussions on various aspects of this work.

Appendix A. Auxiliary lemmata.

Lemma A.1. *Let $A, B > 0$, $\zeta \in (0,1)$. Then*

$$\frac{1}{A+B} \leqslant \frac{1}{A^{\zeta} B^{1-\zeta}}$$

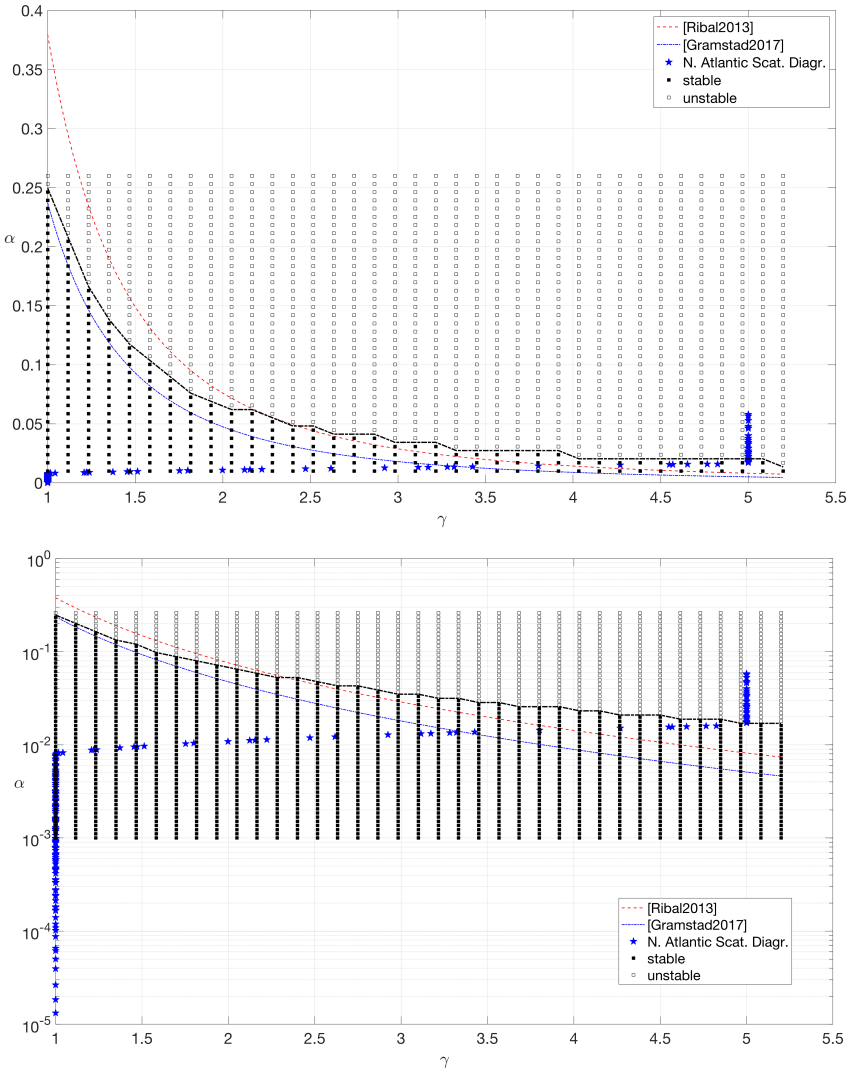

FIGURE 5. A number of points on the (γ, α) plane are tested for stability of the corresponding JONSWAP spectrum, cf. equation 52. α controls the power of the sea state (larger α means larger significant wave height) and γ controls the effective bandwidth (larger γ means more narrowly peaked spectrum). The carrier wavenumber k_0 can easily be seen not to affect the (in)stability of the spectrum. (γ, α) points found to be stable are marked with a full square, while points found to be unstable are marked with an empty square. For reference the proposed separatrices of [29] and [13] are shown (they are of the form $\alpha \cdot \gamma/\beta = C$, where β is the mean wave steepness and $C = 0.77$ [13] or $C = 0.974$ [29]). More details can be found in Section 8. **Top:** Linear scaling in both axes. **Bottom:** Log scaling in the α (vertical) axis.

Proof. The well-known Young's inequality for products implies that, for $a, b > 0$, $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \leqslant a^p + b^q.$$

Now setting $A = a^p$, $B = b^q$ we have

$$A^{1/p} B^{1/q} \leqslant A + B \quad \Rightarrow \quad \frac{1}{A+B} \leqslant \frac{1}{A^{1/p} B^{1/q}}.$$

By setting $\zeta = 1/p$ and observing that $1/q = 1 - 1/p = 1 - \zeta$ the conclusion follows. □

Lemma A.2. *Let $\widetilde{h}(X, s)$ be as in equation 38. Then*

$$\lim_{\rho \to \infty} \sup_{\substack{\mathrm{Re}(\omega) > 0 \\ |\omega| > \rho}} |\widetilde{h}(X, \omega)| = 0.$$

Proof. Recall that $P \in \mathcal{S}(\mathbb{R})$ is of compact support. Hence by construction $XD_X P(k) = P(k + \frac{X}{2}) - P(k - \frac{X}{2})$ is also of compact support for each $X \in \mathbb{R}$. Let $M = M(X)$ be such that $supp\, XD_X P \subseteq [-M, M]$. Then for all ρ large enough we have

$$\begin{aligned} G(\rho) := \sup_{\substack{\mathrm{Re}(\omega) > 0 \\ |\omega| > \rho}} |\widetilde{h}(X, \omega)| &\leqslant \sup_{\substack{\mathrm{Re}(\omega) > 0 \\ |\omega| > \rho}} |q| \int_{\mathbb{R}} \frac{|XD_X P(k)|}{|\omega - 4\pi^2 ipXk|} dk \\ &= |q| \sup_{\substack{\mathrm{Re}(\omega) > 0 \\ |\omega| > \rho}} \int_{-M}^{M} \frac{|XD_X P(k)|}{|\omega - 4\pi^2 ipXk|} dk \\ &\leqslant |q| \int_{\mathbb{R}} |XD_X P(k)| dk \sup_{\substack{\mathrm{Re}(\omega) > 0 \\ |\omega| > \rho,\ |k| < M}} \frac{1}{|\omega - 4\pi^2 ipXk|}. \end{aligned}$$

Clearly $\lim_{\rho \to \infty} G(\rho) = 0$. □

Theorem A.3 (Conditional integrability of the Hilbert transform)**.** *Let $f \in \mathcal{S}(\mathbb{R})$ be a function of compact support with $\int_t f(t) dt = 0$. Then*

$$\|\mathbb{H}[f]\|_{L^1(\mathbb{R})} < \infty.$$

Proof. Choose an $M > 0$ so that the support of f is contained in $[-M, M]$, i.e. $f(x) = 0\ \forall |x| \geqslant M$. We will also use the "double" interval, $J := [-2M, 2M]$ and its complement $J^c = \mathbb{R} \backslash J$. By an elementary estimate we have

$$\begin{aligned} \|\mathbb{H}[f]\|_{L^1(\mathbb{R})} \leqslant 4M \|\mathbb{H}[f]\|_{L^\infty(\mathbb{R})} + \int_{J^c} |H[f](x)| dx &\leqslant 4CM \|\mathbb{H}[f]\|_{H^1(\mathbb{R})} \\ &+ \int_{J^c} |H[f](x)| dx \end{aligned}$$

where C is the constant of the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Moreover, using the fact that $\int_{\mathbb{R}} f(t) dt = 0$ we have

$$\begin{aligned} I := \int_{J^c} |H[f](x)| dx &= \frac{1}{\pi} \int_{J^c} \Big| \int_{\mathbb{R}} \frac{f(t)}{x - t} dt \Big| dx = \frac{1}{\pi} \int_{J^c} \Big| \int_{\mathbb{R}} \left(\frac{f(t)}{x - t} - \frac{f(t)}{x} \right) dt \Big| dx = \\ &= \frac{1}{\pi} \int_{J^c} \Big| \int_{\mathbb{R}} f(t) \left(\frac{1}{x - t} - \frac{1}{x} \right) dt \Big| dx = \frac{1}{\pi} \int_{J^c} \Big| \int_{-M}^{M} f(t) \frac{t}{x(x - t)} dt \Big| dx, \end{aligned}$$

where in the last step we also used the fact that f is supported inside $[-M, M]$. Now observe that for any $x \notin [-2M, 2M]$ and $t \in [-M, M]$

$$|t| \leqslant |x-t| \quad \Rightarrow \quad |x| = |x+t-t| \leqslant 2|x-t| \quad \Rightarrow \quad \frac{1}{|x-t|} \leqslant \frac{2}{|x|}.$$

Hence

$$\left|\frac{t}{x(x-t)}\right| \leqslant \frac{2M}{x^2} \qquad \Rightarrow \qquad I \leqslant \frac{2M}{\pi}\int_{J^c}\frac{1}{x^2}dx\int_{-M}^{M}|f(t)|dt < \infty.$$

□

Appendix B. Derivation of the Alber equation.

Remark B.1. Technically, the Alber equation does not govern any moments of solutions of NLS. It is derived heuristically, assuming the existence of a stochastic solution for the NLS with a certain kind of autocorrelation function and then applying a Gaussian closure to the resulting infinite moment hierarchy. So while, in certain situations, it may well turn out to be a reasonable approximation for certain second moments of solutions of the NLS equation, we don't study such an approximation in this paper.

In what follows in this Section we describe systematically the steps for the heuristic derivation of the Alber equation from the NLS equation. In particular, we use exact properties of certain Gaussian processes which are natural in the linear theory of water waves in order to motivate and justify the Gaussian closure used.

It is important to note that other equations of a similar character can be derived using different assumptions, cf. e.g. [2, 33], and the results of this paper could motivate analogous advances for those equations as well.

To explain the derivation of the Alber equation 1 as a second moment of the NLS 2, first consider the algebraic (deterministic) second moment: denoting

$$R_1(\alpha, \beta, t) := u(\alpha, t)\overline{u}(\beta, t),$$

a straightforward computation leads to

$$i\partial_t R_1 + \frac{p}{2}\left(\Delta_\alpha - \Delta_\beta\right)R_1 + \frac{q}{2}R_1(\alpha,\beta,t)\left[R_1(\alpha,\alpha) - R_1(\beta,\beta)\right] = 0 \tag{57}$$

for the evolution in time of R_1. Thus, despite taking a second moment of a nonlinear equation, the *exact algebraic moment closure*

$$|u(\alpha,t)|^2 u(\alpha,t)\overline{u}(\beta,t) = R_1(\alpha,\alpha,t)R_1(\alpha,\beta,t)$$

allows one to have a closed, exact second moment equation. The same equation is called the "infinite system of fermions" in statistical physics [9]. Now consider the stochastic second moment,

$$R(\alpha,\beta,t) := \mathrm{E}[u(\alpha,t)\overline{u}(\beta,t)].$$

Obviously now the algebraic closure is not enough, as $\mathrm{E}[|u(\alpha,t)|^2 u(\alpha,t)\overline{u}(\beta,t)]$ is a fourth order stochastic moment, and not exactly expressible in terms of second order moments in general. However, for Gaussian processes (under additional assumptions described below) it can be seen that

$$\mathrm{E}[|u(\alpha,t)|^2 u(\alpha,t)\overline{u}(\beta,t)] = 2R(\alpha,\alpha,t)R(\alpha,\beta,t). \tag{58}$$

This is reminiscent of the well known real-valued Isserlis Theorem; the difference is that here u is complex valued (and the factor 2 is an artifact of the complex-valuedness of u). So the Alber equation 1 and the deterministic Wigner transform of the Schrödinger equation 2 differ only in terms of this factor of 2.

The precise result we invoke here can be summarized as follows:

Observation B.1 (A complex Isserlis theorem)**.** *A moment closure result is proved in* [27]*, and a special case of it is the following:*

Let $z(x)$ be a Gaussian, zero-mean, stationary process with the additional property that

$$E[u(x)u(x')] = 0 \qquad \forall x, x' \in \mathbb{R}. \tag{59}$$

Then

$$E[\overline{z(x_1)z(x_2)}z(x_3)z(x_4)] = \\ = E[\overline{z(x_1)}z(x_3)]E[\overline{z(x_2)}z(x_4)] + E[\overline{z(x_2)}z(x_3)]E[\overline{z(x_1)}z(x_4)].$$

This result directly implies the closure relation

$$E[\overline{u(\alpha,t)u(\beta,t)}u(\alpha,t)u(\alpha,t)] = 2E[\overline{u(\alpha,t)}u(\alpha,t)]E[\overline{u(\beta,t)}u(\alpha,t)], \tag{60}$$

which is exactly equation 58.

Moreover, the condition 59 is equivalent to circular symmetry*, i.e. to the condition that*

$$\{e^{i\theta}u(x)\}_{\theta\in[0,2\pi)} \text{ are identically distributed for all } \theta \in [0, 2\pi) \tag{61}$$

by virtue of a result by Grettenberg [35].

Remark B.2 (Physical meaning of the Gaussian closure)**.** Assuming that, for each t_0 the wave envelope $u(x, t_0)$ is a Gaussian process, with mean zero, stationary in x (i.e. spatially homogeneous) and gauge invariant, $e^{i\theta}u(x, t_0) \sim u(x, t_0)$, is in line with standard modelling assumptions for linearized ocean waves [22]. In other words, the Gaussian moment closure of equation 58 can be thought of as a linearization of the probability structure of the wave envelope.

By using the Gaussian closure 58 we see that $R(\alpha, \beta, t)$ satisfies the equation

$$i\partial_t R + \frac{p}{2}\left(\Delta_\alpha - \Delta_\beta\right) R + qR(\alpha,\beta,t)\left[R(\alpha,\alpha) - R(\beta,\beta)\right] = 0, \tag{62}$$

which is structurally the same as the infinite system of fermions, the only difference being an effective doubling of the coupling constant, $q/2 \mapsto q$. Introducing the assumption

$$R(\alpha, \beta, t) = \Gamma(\alpha - \beta) + \epsilon\rho(\alpha, \beta, t),$$

we postulate that R is in leading order homogeneous in space, and we set up an initial value problem for the inhomogeneity $\rho(\alpha, \beta, t)$,

$$i\partial_t\rho + \tfrac{p}{2}\left(\Delta_\alpha - \Delta_\beta\right)\rho + q\left[\Gamma(\alpha-\beta) + \epsilon\rho(\alpha,\beta)\right]\left[\rho(\alpha,\alpha) - \rho(\beta,\beta)\right] = 0. \tag{63}$$

Now denote $\mathcal{R}$ be the rotation operator on phase-space

$$\mathcal{R}[f(x, y)] := f(x + \frac{y}{2}, x - \frac{y}{2}), \tag{64}$$

and consider the *average Wigner transform of the wave envelope* [3, 5]

$$W(x,k,t) = \int_{\mathbb{R}^d} e^{-2\pi iky}\,\mathrm{E}\left[u(x+\frac{y}{2},t)\overline{u(x-\frac{y}{2},t)}\right]dy = \mathcal{F}_{y\to k}\mathcal{R}[R(x,y,t)] = \\ = \mathcal{F}_{y\to k}[\Gamma(y) + \epsilon\rho(x+\frac{y}{2}, x-\frac{y}{2}, t)] = P(k) + \epsilon w(x,k,t). \tag{65}$$

Then the Alber equation 1 is the equation for $w(x,k,t)$, i.e. it results by applying $\mathcal{F}_{y\to k}\mathcal{R}$ to equation 63.

So finally the relation between the unknown of the Alber equation, $w(x,k,t)$, and the wave envelope, $u(x,t)$, is

$$\mathcal{F}_{y\to k}\,\mathrm{E}[u(x+\frac{y}{2},t)\overline{u}(x-\frac{y}{2},t)] \approx P(k)+\epsilon w(x,k,t),$$

where the quality of the approximation rests crucially on how accurate the Gaussian closure is.

Moreover, if $\int_{\mathbb{R}^{2d}} w_0(x,k)dxdk = 0$ we have just an inhomogeneous redistribution of the energy of the homogeneous sea state, while if $\int_{\mathbb{R}^{2d}} w_0(x,k)dxdk > 0$ we have a wave-train of finite energy interacting with a homogeneous sea state of infinite energy.

Appendix C. Background results on Laplace and Hilbert transforms.

Theorem C.1 (Regularity of the Hilbert & signal transforms)**.** *Let $1<p<\infty$. Then there exist constants $C=C(p)$ such that*

$$\|\mathbb{H}[u]\|_{L^p(\mathbb{R})} \leqslant C\|u\|_{L^p(\mathbb{R})}, \qquad \|\mathbb{S}[u]\|_{L^p(\mathbb{R})} \leqslant (1+C)\|u\|_{L^p(\mathbb{R})}.$$

Moreover, $C(2)=1$ and for any $s\in\mathbb{N}$,

$$\|\mathbb{H}[u]\|_{H^s(\mathbb{R})} = \|u\|_{H^s(\mathbb{R})}, \qquad \|\mathbb{S}[u]\|_{H^s(\mathbb{R})} \leqslant 2\|u\|_{H^s(\mathbb{R})}.$$

Combining this with the Sobolev embedding $H^1(\mathbb{R})\hookrightarrow C^0(\mathbb{R})$ it follows that

$$u\in H^1(\mathbb{R}) \quad\Rightarrow\quad \mathbb{H}[u],\mathbb{S}[u]\in C^0(\mathbb{R}).$$

Theorem C.2 (Sokhotski-Plemelj formula)**.** *For $u\in C(\mathbb{R})\cap L^1(\mathbb{R})$ and for any $s,c\in\mathbb{R}$*

$$\lim_{\eta\to 0^+}\mathbb{H}[u]\Big(\frac{s-i\eta}{c}\Big) = \mathbb{S}[u]\Big(\frac{s}{c}\Big).$$

Theorem C.3 (Inverse Laplace transform, open half-plane)**.** *Let $F(\omega)$ be a bounded analytic function on an open right half-plane, $\omega\in\Pi(M):=\{\mathrm{Re}\,z>M\}$. Assume moreover that the limit $F_{M^+}(b):=\lim_{\varepsilon\to 0^+}F(M+\varepsilon+ib)$ exists for all $b\in\mathbb{R}$ and is a continuous function in b. Moreover assume that*

$$\lim_{\rho\to+\infty}\sup_{\substack{\omega\in\Pi(M)\\ |\omega|>\rho}}|F(\omega)|=0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty}|F_{M^+}(s)|ds<\infty. \tag{66}$$

Then

$$F(\omega)=\mathcal{L}_{t\to\omega}[f(t)] \qquad \text{where} \qquad f(t)=\frac{e^{Mt}}{2\pi}\int_{-\infty}^{+\infty}e^{ist}F_{M^+}(s)ds,$$

i.e.

$$\mathcal{L}^{-1}_{\omega\to t}[F]=\frac{e^{Mt}}{2\pi}\int_{-\infty}^{+\infty}e^{ist}F_{M^+}(s)ds.$$

Appendix D. **Moments and Derivatives of the Alber-Fourier equation.** Denote

$$\begin{aligned} L[P_1 \neg P_2; m] &:= \Big[P_1\Big(k - \frac{X}{2}\Big) - P_2\Big(k + \frac{X}{2}\Big)\Big] m(X,t) \\ N[m; f_1 \neg f_2] &:= \int_s m(s,t)\Big[f_1\Big(X - s, k - \frac{s}{2}, t\Big) - f_2\Big(X - s, k + \frac{s}{2}, t\Big)\Big] ds. \end{aligned} \tag{67}$$

The nonlinearity $\mathbb{B}[m, f]$ defined in equation 23 is comprised of

$$\mathbb{B}[m, f] = iqL[P \neg P; m] + \epsilon iqN[m; f \neg f].$$

Lemma D.1. *For any multi-indices $\alpha, \beta, \gamma, \delta \in (\mathbb{N} \cup \{0\})^d$ we have the following relations*

$$\begin{aligned} X^\alpha L[P_1 \neg P_2; m] &= L[P_1 \neg P_2; X^\alpha m], \\ k^\beta L[P_1 \neg P_2; m] &= \sum_{0 \leqslant \beta' \leqslant \beta} \binom{\beta}{\beta'} L[k^{\beta - \beta'} P_1 \neg (-1)^{\beta'} k^{\beta - \beta'} P_2; (\frac{X}{2})^{\beta'} m], \\ \partial_X^\gamma L[P_1 \neg P_2; m] &= \sum_{0 \leqslant \gamma' \leqslant \gamma} \binom{\gamma}{\gamma'} L[(-1/2)^{\gamma - \gamma'} \nabla^{\gamma - \gamma'} P_1 \neg (1/2)^{\gamma - \gamma'} \nabla^{\gamma - \gamma'} P_2; \partial_X^{\gamma'} m], \\ \partial_k^\delta L[P_1 \neg P_2; m] &= L[\nabla^\delta P_1 \neg \nabla^\delta P_2; m], \end{aligned}$$

and

$$\begin{aligned} X^\alpha N[m; f_1 \neg f_2] &= \sum_{0 \leqslant \alpha' \leqslant \alpha} \binom{\alpha}{\alpha'} N[X^{\alpha - \alpha'} m; X^{\alpha'} f_1 \neg X^{\alpha'} f_2], \\ k^\beta N[m; f_1 \neg f_2] &= \sum_{0 \leqslant \beta' \leqslant \beta} \binom{\beta}{\beta'} N\Big[\Big(\frac{X}{2}\Big)^{\beta - \beta'} m; k^{\beta'} f_1 \neg (-1)^{\beta - \beta'} k^{\beta'} f_2\Big], \\ \partial_X^\gamma N[m; f_1 \neg f_2] &= N[m; \partial_X^\gamma f_1 \neg \partial_X^\gamma f_2], \\ \partial_k^\delta N[m; f_1 \neg f_2] &= N[m; \partial_k^\delta f_1 \neg \partial_k^\delta f_2]. \end{aligned}$$

Moreover,

$$\begin{aligned} X^\alpha k^\beta \partial_X^\gamma \partial_k^\delta \big(k \cdot X f\big) = k \cdot X \big(X^\alpha k^\beta \partial_X^\gamma \partial_k^\delta f\big) \\ + X^\alpha k^\beta \sum_{\substack{0 \leqslant \gamma' < \gamma \\ 0 \leqslant \delta' < \delta}} \binom{\gamma}{\gamma'} \binom{\delta}{\delta'} \big(\partial_X^{\gamma'} \partial_k^{\delta'} f\big) \big(\partial_X^{\gamma - \gamma'} \partial_k^{\delta - \delta'} X \cdot k\big). \end{aligned}$$

The proof follows from direct computations using the definition of $L[P_1 \neg P_2; m]$ and $N[m; f_1 \neg f_2]$.

By applying the operator $X^\alpha k^\beta \partial_X^\gamma \partial_k^\delta$ to equation 22 and commuting according to Lemma D.1 one obtains equation 27 with right hand side

$$\mathbb{B}^{(\alpha,\beta,\gamma,\delta)}[f] = - \sum_{\substack{0\leqslant\gamma'\leqslant\gamma\\ 0\leqslant\delta'\leqslant\delta\\ |\gamma'+\delta'|<|\gamma+\delta|}} \binom{\gamma}{\gamma'}\binom{\delta}{\delta'}\big(\partial_X^{\gamma-\gamma'}\partial_k^{\delta-\delta'} X\cdot k\big) f^{\alpha,\beta,\gamma',\delta'}$$

$$- qi \sum_{\substack{0\leqslant\beta'\leqslant\beta\\ 0\leqslant\gamma'\leqslant\gamma}} (\frac{1}{2})^{|\gamma-\gamma'|+|\beta'|}\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}$$

$$\times L[k^{\beta-\beta'}(-1)^{|\gamma-\gamma'|}\nabla^{\gamma-\gamma'+\delta}P \multimap k^{\beta-\beta'}\nabla^{\gamma-\gamma'+\delta}P\,;\, \int_k f^{\alpha+\beta',0,\gamma',0}dk]$$

$$- \epsilon qi \sum_{\substack{0\leqslant\alpha'\leqslant\alpha\\ 0\leqslant\beta'\leqslant\beta}} (\frac{1}{2})^{|\beta-\beta'|}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}$$

$$\times N[\int_k f^{\alpha-\alpha'+\beta-\beta',0,0,0}dk\,;\, f^{\alpha',\beta',\gamma,\delta} \multimap (-1)^{|\beta-\beta'|} f^{\alpha',\beta',\gamma,\delta}].$$

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Received June 2019; revised January 2020.

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