Extreme properties of impact-induced vertical bending moments

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ABSTRACT
Development of a reduced-order mathematical model suitable for studies of extreme properties of impact-induced vertical bending moments (VBM) is described. Extreme values of impact-induced VBM are results of rare events that are difficult to simulate in time-domain, since the computational cost of time-domain simulations with sufficient fidelity is too high for regular engineering practice. This is known as a "problem of rarity" in dynamic stability. The objective is to formulate a qualitative reduced-order model that will reveal the structure of the tail of the impact-induced VBM distribution. Then the "correct" tail can be fitted with reasonable volume of sufficient fidelity simulation data. Zero-upcrossing of relative motion of a reference point is considered as a precursor of a possible impact event. As the first derivative of relative motion at the instants of impact is known, the impact force can be computed. To a first approximation this is done with a wedge approximation, using the estimated reentry velocity and added mass of a station, containing the reference point. The structural response is then obtained in terms of an elastic beam model and a normal mode expansion. The distribution of a maximum of impact-induced VBM (and its timing since the impact event) is computed numerically using known distribution of the velocities at the instants of crossing.
Keywords: Vertical Bending Moment; Extreme Events; Impact due to slamming

1 INTRODUCTION

The statistics of the vertical bending moment (VBM) is a critical piece of information for the fatigue analysis of ships. However, its computation is highly non-trivial due to the highly non-Gaussian structure which is a direct manifestation of the non-linear dynamics and their complex interaction with the stochastic waves. Several factors contribute on the formation of the non-Gaussian characteristics: the nonlinear Froude-Krylov (FK) interactions, deck submergence effects, but also structural vibrations induced due to slamming events. The non-Gaussian structure induced by the Froude-Krylov terms has recently been analytically studied in [7]. Here we focus on understanding the effects due to impact-induced VBM caused by slamming events of the hull and the resulted whipping. These can be quite severe and can play an important role on the PDF of the VBM (see Fig. 1). Most importantly, they have much higher frequency compared with FK VBM, contributing significantly to the fatigue lifetime.

Here analytical approximations for the PDF describing the VBM acting on the ship due to whipping effects are derived. This analysis complements efforts focused on the analysis of VBM statistics induced by Froude-Krylov nonlinear interactions with water waves [7]. In this the effect of slamming events (and the induced whipping loads) is the focus. First, the slamming events are formulated as an up-crossing problem. From the ship characteristics and the statistics for pitch motion closed form expressions are derived for the frequency of slamming events, as well as their velocity and duration. This information is then transformed into an impact force acting on the hull. The next step involves the analysis of the whipping response. This is done from a standard beam approximation for the ship structure and maintaining only the first normal mode. The result is an analytical approximation for the impact-induced VBM at any location of the ship, in terms of the hull geometry and the water waves spectrum. The form of this PDF will be discussed and connections with direct numerical simulations will be provided.

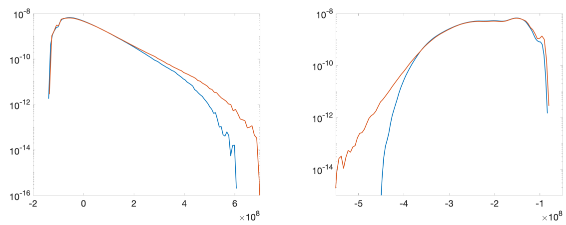

Figure 1: Vertical bending moment peak statistics (left-positive; right-negative). Blue curves indicate VBM due to quasi-static (Froude-Krylov) interactions with waves. Red curves describe total VBM, i.e. including peaks due to slamming.

2 SETUP OF IMPACTS

Given the close-to-Gaussian statistical character of the pitch motion, confirmed by LAMP simulations [7], the linearized pitch model is applied. More detailed models containing additional degrees-of-freedom, such as heave, may be considered.

$$I\ddot{\theta} + c\dot{\theta} + K\theta = M_{FK}(t;\zeta), \qquad (1)$$

where $M_{FK}(t;\zeta)$ is a random excitation moment caused by the random waves, and θ is the pitch angle (positive θ is bow down). Also, the wave elevation at FP given by $h_{FP}(t;\zeta)$ is included. The slamming events are assumed to occur when the hull at forward perpendicular (FP) touches the water surface with non-zero velocity. In Fig. 2 this event is defined through the distance of the water plane from the wave elevation at FP, denoted as, $z(t)$:

$$z(t;\zeta) = -\theta(t;\zeta)\frac{L}{2} - h_{FP}(t;\zeta), \qquad (2)$$

A possible extension that includes heave effects will model this parameter as a stochastic process. Denoting as z_h the constant (signed) distance from the water plane to the keel, the hull touches the water when $z = z_h < 0$. Values of z greater than z_h correspond to cases when the hull is out of the water, while values smaller than z_h imply hull in the water.

Under this formulation, a slamming event is defined as the down-crossing event

$$z = z_h \quad \text{and} \quad \dot{z} < 0. \qquad (3)$$

The statistics of the stochastic process, z can be computed in the context of a Gaussian approximation with short numerical simulations. Specifically, σ_z and $\sigma_{\dot{z}}$ are assumed known.

3 STATISTICS OF SLAMMING EVENTS

Next, the conditional statistics of the slamming velocity, $p(\dot{z}|z = z_h)$, are expressed, as well as, the average frequency of slamming events, $n(z_h)$. The structural response analysis of the hull is analyzed when these slamming events occur.

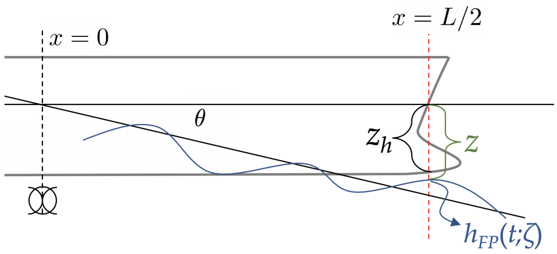

Figure 2: Slamming event geometry

The frequency of down-crossing events for a zero-mean, stationary and ergodic stochastic process, $z(t;\zeta)$, is given by the Rice formula [6]

$$\bar{n}(z_h) = \int_0^\infty \dot{z} p_{z\dot{z}}(z_h, \dot{z}) d\dot{z}. \tag{4}$$

For the case of a Gaussian stochastic process the above formula takes the simpler form

$$\bar{n}(z_h) = \frac{1}{2\pi} \frac{\sigma_{\dot{z}}}{\sigma_z} e^{-\frac{z_h^2}{2\sigma_z^2}}. \tag{5}$$

In addition, the conditional statistics of the relative velocity between the hull and the wave when the slamming event occurs (denoted as slamming velocity) is given by the Rayleigh distribution [3]

$$p(\dot{z}|z = z_h) = -\frac{\dot{z}}{\sigma_{\dot{z}}} e^{-\frac{\dot{z}^2}{2\sigma_{\dot{z}}^2}}, \quad \dot{z} < 0, \tag{6}$$

where $\sigma_{\dot{z}}$ is the corresponding variance.

4 STRUCTURAL RESPONSE

The structural response and induced vertical bending moments due to slamming loads are analyzed. Following [1] the beam model is employed:

$$M_B \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = q(x,t), \tag{7}$$

where M_B is the mass per unit length, EI is the elastic modulus times the moment of inertia of the cross section, q is the slamming loads and w is the vertical displacement of the beam. The bending moment due to slamming loads at each point of the beam is given by

$$M_s = -EI \frac{\partial^2 w}{\partial x^2}. \tag{8}$$

For boundary conditions, zero displacements and zero bending moments are assumed as this is a good approximation for relevant cases [1], i.e.

$$w\left(\pm\frac{L}{2}, t\right) = \frac{\partial^2 w}{\partial x^2}\left(\pm\frac{L}{2}, t\right) = 0. \tag{9}$$

With these boundary conditions, the normal modes are:

$$\Psi_n(x) = \cos\left(\frac{M_B \omega_n^2}{EI} x\right), \tag{10}$$

and associated eigenvalue equations:

$$\frac{M_B \omega_{n+1}^2}{EI} L/2 = \pi/2 + n\pi, \; n = 0, 1, 2, ... \tag{11}$$

where ω_n are the natural frequencies. Following experiments, the first mode, that is, $n = 1$ is the one that is typically dominating. Therefore, we will be studying the first (normalized) mode

$$\Psi_1(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right), \omega_1 = \left(\frac{EI}{M_B}\right)^{\frac{1}{2}} \left(\frac{\pi}{L}\right)^2. \tag{12}$$

In this way the projected equation is obtained:

$$\ddot{a}_1 + \omega_1^2 a_1 = \frac{1}{M_B} Q(t), \tag{13}$$

where $Q(t) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) q(x,t) dx$. To the above, damping is added and the model is obtained as:

$$\ddot{a}_1 + 2\zeta\omega_1 \dot{a}_1 + \omega_1^2 a_1 = \frac{Q(t)}{M_B}. \tag{14}$$

From which it follows

$$a_1(t) = \frac{1}{\omega_{D_1} M_B} \int_0^t Q(\tau) e^{-\zeta\omega_1(t-\tau)} \sin(\omega_{D_1}(t-\tau)) d\tau, \tag{15}$$

where $\omega_{D_1} = \omega_1\sqrt{1-\zeta^2}$. From this solution, the induced vertical bending moment at location x is expressed as:

$$M_s(x,t) = EI \left(\frac{\pi}{L}\right)^2 \cos\left(\frac{\pi x}{L}\right) a_1(t). \tag{16}$$

Under an impulsive excitation hypothesis where the load acts instantaneously at $t = t_0$:

$$q(x,t) = r(x)\delta(t - t_0). \tag{17}$$

Substituting, the following is obtained:

$$a_1(t) = \frac{R}{\omega_{D_1} M_B} e^{-\zeta\omega_1(t-t_0)} \sin(\omega_{D_1}(t-t_0)), \; t \geq t_0, \tag{18}$$

where $R = \int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi y}{L}\right) r(y)dy$. In this simple model, R can be modeled as a function of the relative velocity of the bow (at the moment of slamming) and the moment of impact, t_0, as a Poisson process with frequency parameter $\bar{n}$.

5 STATISTICS OF THE IMPACT FORCE

To compute the impact forces due to slamming, the time-dependent added mass acting on each cross-section is given by

$$A(x,\zeta) = \frac{1}{2}\pi^3\rho\left(1 - \frac{\beta(x,\zeta)}{2\pi}\right)^2\left(\frac{b(x,\zeta)}{2}\right)^2, \quad (19)$$

where ζ is the local draft taking into account the local wave elevation, $\zeta = -\frac{L}{2}\theta - h(x,t)$, β is the local deadrise angle, b is the local 1/2 beam, and ρ is the water density. The sectional impact force is computed as the downward relative vertical velocity of the section times the time-derivative of the sectional added mass:

$$q(x,t) = -\dot{\zeta}\frac{dA(x,\zeta)}{dt}\delta(t-t_0). \quad (20)$$

As a first order approximation, that the deadrise angle is assumed constant over ζ, while the local half beam changes linearly with z, i.e. $b(x,\zeta) = b_0(x) + b_1(x)\zeta$. With these assumptions the following can be written:

$$\frac{dA(x,\zeta)}{dt} = \gamma_0(x)\dot{\zeta}\zeta + \gamma_1(x)\dot{\zeta}, \quad (21)$$

where $\gamma_0(x) = \pi^3\rho\left(1 - \frac{\beta_0(x)}{2\pi}\right)^2\frac{b_1(x)^2}{2}$, and $\gamma_1(x) = \pi^3\rho\left(1 - \frac{\beta_0(x)}{2\pi}\right)^2\frac{b_0(x)b_1(x)}{2}$.

Therefore,

$$q(x,t) = -(\gamma_0(x)\dot{\zeta}^2\zeta + \gamma_1(x)\dot{\zeta}^2)\delta(t-t_0). \quad (22)$$

Slamming loads are assumed to occur around the bow area, i.e. for $x > x_s$, where x_s defines the point over which slamming loads are significant. To this end ζ at the FP location is approximated:

$$\zeta \simeq z = -\theta\frac{L}{2} - h_{FP}. \quad (23)$$

In addition, slamming events occur for $z = z_h$. Therefore,

$$q(x,t) = -\dot{z}^2\left(\gamma_0'(x)z_h + \gamma_1'(x)\right)\delta(t-t_0), \quad (24)$$

where $\gamma_i'(x) = h(x-x_s)\gamma_i(x), i = 0,1$, with $h(x)$ being the step function. While the instantaneous impact model provides a quadratic expression in terms of the impact velocity, a linear term is adopted, to account for finite-time effects and other factors that have not been included in our analysis. Based on this approximation the following is obtained

$$R(\dot{z}) = -c_0\dot{z}^2 - c_1\dot{z}. \quad (25)$$

Given the Rayleigh approximation of the impact velocity due to slamming, an analytical approximation for the PDF can be obtained, $p_R(r)$.

In Figure 3 the PDF for different values of the parameters associated with the slamming force is plotted. To to emphasize the heavy tail character inherited by the quadratic character of the hydrodynamic force with respect to the impact velocity is important.

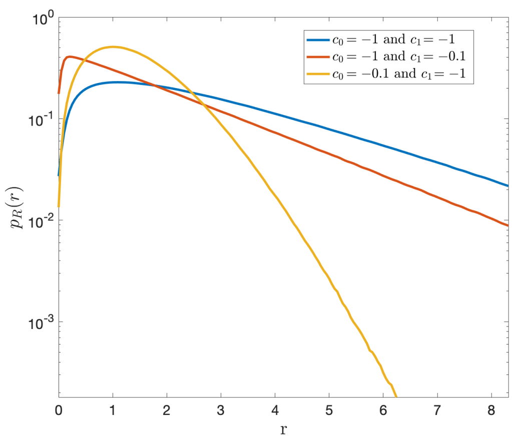

Figure 3: Pdf for the impulsive force during slamming for different values of the hydrodynamic parameters, c_0 and c_1. For all cases $\sigma_{\dot{z}} = 1$.

6 STATISTICS OF VBM

To characterize the pdf of the vertical bending moment so that the effects of slamming is included, the probabilistic decomposition-synthesis method [5] is applied. In this context, the probabilistic response is expressed as the sum of the conditional statistics corresponding to rare events (slamming), as well as the conditional statistics corresponding to regular events (Froude-Krylov loads). An important assumption is made here: *specifically, the impact-induced VBM dominates over the FK VBM whenever an impact occurs*, i.e. the FK-induced VBM is negligible in magnitude compared with the impact induced VBM. In this case (p_T being the PDF for the total VBM, p_I is the impact-induced, and p_{FK} the FK VBM):

$$p_T(M) = p_I(M|\text{Impact}, |M| > \gamma)\mathbb{P}_r + p_{FK}(M|FK)(1 - \mathbb{P}_r). \quad (26)$$

where $\mathbb{P}_r$ is the overall impact probability. This is defined as the probability of the VBM, M, exceeding a threshold γ due to a slamming event

$$\mathbb{P}_r = \frac{1}{T} \int_{t\in[0,T]} 1(|M| > \gamma, \text{Impact})dt, \quad (27)$$

where $1(\centerdot)$ is the indicator function. The rare event probability measures the total rare event duration. Moreover, the conditional probability $p_I(M||M| > \gamma, \text{Impact})$ is the pdf of the VBM given that a slamming event and exceeds a certain critical threshold γ. On the other hand, $p_{FK}(M|FK)$ denotes the conditional probability of the VBM in the absence of slamming events. This has already been analyzed in [7].

The focus here will be on identifying the conditional pdf $p_I(M|\text{Impact}, |M| > \gamma)$ and the probability of extreme VBM values due to slamming, $\mathbb{P}_r$. The beam response is assumed to be highly oscillatory, i.e. underdamped ($\zeta \ll 1$). In this case, to focus on the PDF for the local maxima of the VBM rather than the VBM value itself, is meaningful. This is only the case for oscillatory events induced by slamming which 'live' in very small time scales compared with the VBM induced by irregular waves, which, in general, is a slow process. The analytical approximation (18) is applied. The local maxima will have the form of the envelope [2]:

$$\tilde{M}_s(x,t) = Ke^{-\zeta\omega_1(t-t_0)}, \quad t \geq t_0, \quad (28)$$

where,

$$K(\dot{z}) = \sqrt{\frac{EI}{M_B}} \frac{R(\dot{z})}{\sqrt{1-\zeta^2}} \cos\left(\frac{\pi x}{L}\right), \quad (29)$$

is a random variable with known pdf, $p_K(K)$, that depends on the relative velocity statistics at the moment of impact.

6.1 Probability of slamming: $\mathbb{P}_r$

Next, the rare event probability is computed for the total rare event period over a fixed time interval, as defined in eq. (27). This is done by employing an appropriate definition of an extreme event in terms of a threshold value. One possible option is to set an absolute threshold γ. However, in the present context, to set this threshold relative to the value of the local maximum of the extreme event response is more convenient. Specifically, the time duration τ_e, a rare response takes to return back to the background state will be given by the duration starting from the initial impulse event time (t_0) to the point where the response has decayed back to ρ_c of its absolute maximum; here and throughout this paper, $\rho_c = 0.1$. This is a value without tuning. Numerical experiments have shown [2] that the derived approximation is not sensitive to the exact value of ρ_c as long as this has been chosen within reasonable values. In the current context, this means that the rare event duration τ_e is chosen as

$$\tilde{M}_s(x, t_0 + \tau_e) = \rho_c \tilde{M}_s(x, t_0). \quad (30)$$

The above equation is solved with the expression for the envelope and obtain the approximation

$$\tau_e = -\frac{1}{\zeta\omega_1}\log\rho_c. \qquad (31)$$

From this rare event duration, τ_e, the probability of rare events is computed as

$$\mathbb{P}_r = \bar{n}(z_h)\tau_e. \qquad (32)$$

6.2 Conditional PDF for impact VBM

To compute the PDF of the local maxima, for each slamming event, time of any subsequent peak is taken as a random variable that is uniformly distributed between the moment of slamming, t_0, and the end time of slamming oscillation τ_e . Specifically, the following random variable is considered:

$$\tilde{M}_s(x,t) = Ke^{-\zeta\omega_1 t'}, \quad t \geq t_0, \qquad (33)$$

where

$$t' \sim \text{ uniform}(0, \tau_e). \qquad (34)$$

By conditioning on the amplitude of the slamming event, K, the derived distribution is found for the conditional PDF for the local maxima given by (see Appendix B in [2] for a detailed derivation)

$$p_I(M|K) = \frac{h(M-K\rho_c) - h(M-K)}{M\log\rho_c^{-1}}, \qquad (35)$$

Plots of this conditional PDF for various values of K are presented in Figure 4.

Multiplying with the pdf of K (that is easily obtained from the pdf of R and eq. (29)) we have

$$p_I(M) = \int_0^\infty \frac{\pi^2 p_I(M|K) p_R\left(\frac{\pi^2 K}{2L\omega_{D_1}\cos\left(\frac{\pi x}{L}\right)}\right)}{2L\omega_{D_1}\cos\left(\frac{\pi x}{L}\right)} dK. \qquad (36)$$

The shape of the derived PDF for the peaks of the VBM caused by slamming events is shown in Figure 5 for various combinations of the hydrodynamic parameters. The non-uniform

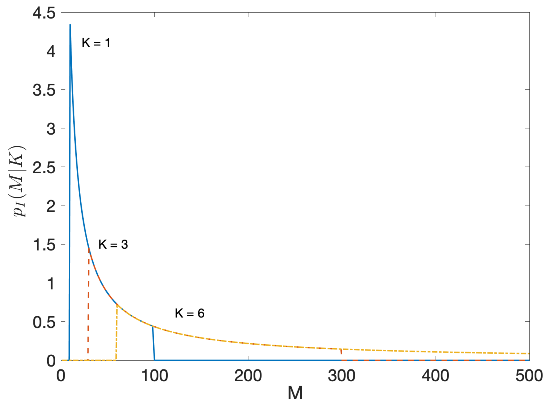

Figure 4: Conditional pdf, $p_I(M|K)$, for various values of K.

character of the tail for the third combination of hydrodynamic parameters is noted. A comparison between the analytical approximation (from optimal parameters where $\frac{c_0}{c_1} = 8.12$) of the PDF and the one obtained with LAMP is shown in Figure 6. The quadratic coefficient, c_0 is dominating over the linear coefficient c_1, which is consistent with the initially derived quadratic form of $R(\dot{z})$ based on first principles (the linear terms was added heuristically to model possible omissions from the original derivation).

7 STATISTICS OF THE TOTAL VBM

The final step of the analysis involves the combination of the two conditional PDFs. This is done through the total probability argument, eq. (26). The PDF for the Froude-Krylov VBM does not refer to local peaks as the waves are slow enough (compared to the fast structural vibrations) for this interaction to be considered quasi-static. On the other hand, the VBM loads induced by slamming events are evolving very quickly and for this reason, the local peaks are considered. The two PDFs can be synthesized and refer to the peaks of the total VBM.

In Figure 7, a direct comparison is presented between the statistics obtained with LAMP

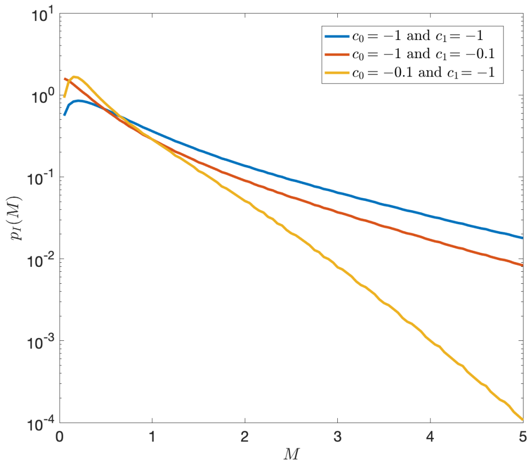

Figure 5: Probability density function for the VBM peaks due to slamming, $p_I(M)$, for $\rho_c = 0.1$ and various values of the hydrodynamic parameters.

and the optimal approximation from the expression (26). LAMP simulations were performed with the ONR Topsides Fared hull (Fig. 8). The analytical expressions is applied for the PDFs and the parameters obtained already in the previous steps and only the parameter, $\mathbb{P}_r$ is optimized.

8 ALTERNATIVE MODEL FOR IMPACT FORCE

An alternative model of the impact forces may also be utilized. This alternative model is expected to be consistent with the model used in LAMP [8]. The idea is to account for the actual geometry of a section. For this model, the velocity of the relative motion is computed with a Slepian model [4], where a second order polynomial approximates the process of relative motion after upcrossing.

$$z(t) = w_U t - \frac{\sigma_z^2}{\sigma_w^2} t^2, \quad w_U > 0. \qquad (37)$$

The relative velocity at the instant of upcrossing $w_U = \dot{z}_U$ follows Rayleigh distribution [3]. Relative velocity during the impact is

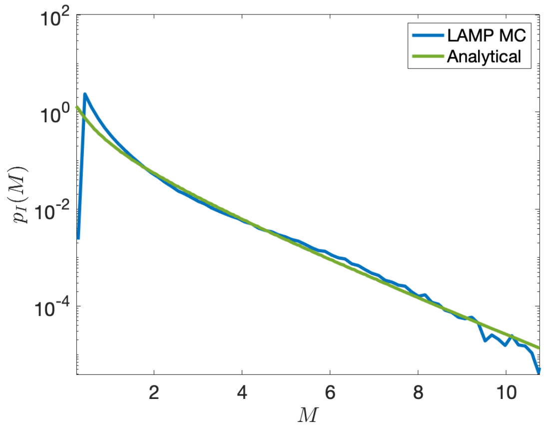

Figure 6: Vertical bending moment associated with slamming loads (positive peaks): Comparison of LAMP results with the analytical approximation (36). Coefficients are optimally tuned ($\frac{c_0}{c_1} = 8.12$).

available from differentiation of (37):

$$w(t) = w_U - \frac{\sigma_z^2}{\sigma_w^2} t, \quad w_U > 0. \qquad (38)$$

The impact force is computed similar to equation (20); however, δ-function is no longer used as the pulse is modeled with rapid change of added mass, evaluated with equation (19) utilizing actual geometry of a section.

$$q(x,t) = -w(t) \frac{dA(x,\zeta)}{dt}. \qquad (39)$$

The relative velocity at the instant of upcrossing is the only random parameter in equation (39). The maximum bending moment caused by the impact can be computed as a deterministic function of a random argument. The calculation follows the same logic, but do not yield semi-analytical PDF (36), but remains numerical. The resulting model may be useful as a sort of an "intermediary" between the "true" reduced-order semi-analytical model and numerical approach [8].

9 SUMMARY AND CONCLUSIONS

Development of reduced-order model for impact-induced vertical bending moment is

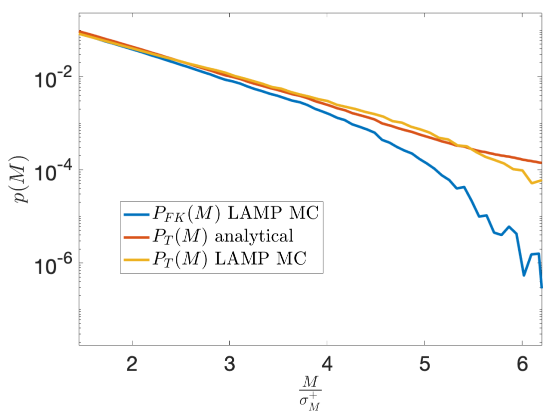

Figure 7: VBM computed with LAMP with (yellow) and without (blue) taking into account the effect of slamming events. The analytical approximation is shown with red color.

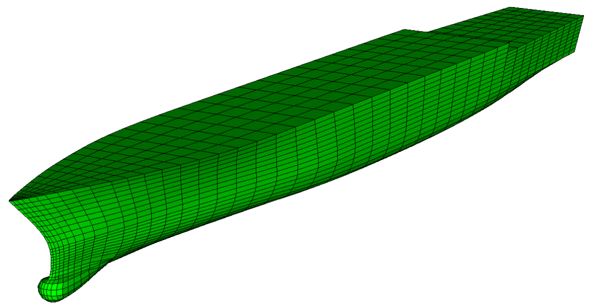

Figure 8: The ONR Topsides Fared hull used for the LAMP Monte-Carlo simulations.

described. The model making use of the following components:

- Upcrossing theory provides information on the velocities at the instants of crossing;
- Impact force is modelled with change of added mass of a section;
- Elastic beam model is used to find the maximum bending moment.

The model yielded semi-analytical expression for PDF of the impact-induced VBM. The problem of combining PDF form impact-induced and wave-induced VBM was briefly considered. Comparison if derived approximate PDF with an outcome of large-volume LAMP simulation has shown a very reasonable agreement, showing a promise of practical applicability. Alternative numerical technique with Slepian model was also explored.

Direction of future work is envisions formalizing the fitting technique and further studying combined extreme VBM induced by waves and whipping.

10 ACKNOWLEDGEMENTS

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